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Dirichlet Forms and Dirichlet Operators for Gibbs Measures of Quantum Unbounded Spin Systems: Essential Self-Adjointness and Log-Sobolev Inequality

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For each $\gamma \in [0, 1]$ we consider the Dirichlet form $\mathscr{E}_\mu^{(\gamma)}$ and the associated Dirichlet operator $H_\mu^{(\gamma)}$ for the Gibbs measure μ of quantum unbounded spin systems interacting via superstable and regular potential. The Gibbs measure μ is related to the Gibbs state of the system via a (functional) Euclidean integral procedure. The configuration space for the spin systems is given by $\Omega := E^{\mathbb{Z}^\nu}$, $E := \{\omega \in C([0, 1]; \mathbb{R}^d)\colon \omega(0) = \omega(1)\}$. We formulate Dirichlet forms in the framework of rigged Hilbert spaces which are related to the space Ω. Under appropriate conditions on the potential, we show that the Dirichlet operator $H_\mu^{(\gamma)}$ is essentially self-adjoint on the domain of smooth cylinder functions. We give sufficient conditions on the potential so that the corresponding Gibbs measure is uniformly log-concave (ULC). This property gives the spectral gap of the Dirichlet operator $H_\mu^{(\gamma)}$ at the lower end of the spectrum. Furthermore, we prove that under the conditions of (ULC), the unique Gibbs measure μ satisfies the log-Sobolev inequality (LS). We use an approximate argument used in the study of the same subjects for loop spaces, which in turn is a modification of the method originally developed by S. Albeverio, Yu. G. Kondratiev, and M. Röckner.

KEY WORDS: Quantum unbounded spin systems; superstable interactions; Gibbs measures; Dirichlet forms; Dirichlet operators; approximate criterion; essential self-adjointness; uniform log-concavity; log-Sobolev inequality.

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1. INTRODUCTION

We consider Dirichlet forms and the associated Dirichlet operators for the Gibbs measures of quantum unbounded spin systems interacting via superstable and regular potentials. The Gibbs measures we deal with can be understood in the sense of Euclidean Gibbs measures associated to the Gibbs states of the systems via a (functional) Euclidean integral procedure. We study the essential self-adjointness problem of Dirichlet operators on a minimum definition of domains and some related properties for Gibbs measures including the L^2-ergodicity and the log-Sobolev inequality for Gibbs measures. This paper is a continuation of ref. 37 in which we have studied the above mentioned subjects for Gibbs measures on loop spaces and also a continuation of our former work[32] in which we have investigated the Dirichlet forms and the associated diffusion processes for Gibbs measures of quantum unbounded spin systems.

The theory of Dirichlet forms on finite dimensional spaces is a well known modern tool in potential theory[21, 44] and quantum mechanics.[4, 53] The extension of the theory to infinite dimensional spaces is more recent (e.g., refs. 2–8, 11–12, 34, 39 and the references therein). In all case the forms are given first on some minimal domains. Most of results then touch upon the problems of the closability of the forms and the construction of corresponding diffusion processes. The uniqueness problem of determining whether a given closable form possessing the contraction property has a unique closed extension has also been discussed in recent years.[13–14, 42–43, 50–52] Clearly the essential self-adjointness of the associated Dirichlet operator implies the uniqueness. In this direction, various conditions for the essential self-adjointness of the Dirichlet operators have been obtained.[5–8, 27, 37, 50–51] The results have been applied to the Dirichlet operators corresponding to Gibbs measures of classical unbounded spin systems.[7–8, 31] Main purpose of this paper is to extend the results in refs. 7, 31, and 37 to quantum unbounded spin systems.

In applications, the presence of log-Sobolev inequality for the Gibbs measures is essential to prove the L^2-ergodicity and the hypercontractivity of the semi-group $T_t = \exp(-H_\mu t)$, $t \geqslant 0$, generated by Dirichlet operator H_μ and it has a wide range of applications.[20] The log-Sobolev inequality was first proven by Gross,[23] and then extended in many directions.[7, 9, 20, 24, 31, 37, 47–49, 56–59]

The quantum unbounded spin systems can be viewed as a model for the quantum anharmonic crystals[19] and is closely related to lattice field theory with continuous times.[1] There have been extensive studies on the systems (see refs. 9, 35 and references therein). The existence of infinite volume limit Gibbs measures has been established in refs. 15 and 35.

Method of cluster expansions has been developed in refs. 1, 26, 36 and existence of phase transitions for ferromagnetic type interactions has been proved by using reflection positivity methods.[16, 25] Stochastic dynamics and Gibbs measures for the model have been also constructed by stochastic quantization method.[10] Recently, the uniqueness of Gibbs measures for certain interaction systems was shown by using Dobrushin's uniqueness criterion.[9]

Let us briefly describe the contents of this paper. We consider quantum unbounded spin sytems on the ν-dimensional lattice space $\mathbb{Z}^{\nu}$.[9, 35–36] For each finite subset $\Lambda \subset \mathbb{Z}^{\nu}$, the local Hamiltonians are given by

$$\begin{aligned} H_\Lambda &= -\tfrac{1}{2} \sum_{i \in \Lambda} \Delta_i + V(x_\Lambda) \\ V(x_\Lambda) &= \sum_{i \in \Lambda} P(x_i) + \sum_{\{i,j\} \subset \Lambda} U(x_i, x_j; |i-j|) \end{aligned} \tag{1.1}$$

where Δ_i is the Laplacian operator on $L^2(\mathbb{R}^d)$ for each $i \in \Lambda$, and $P(x)$ and $U(x, y; r)$, $r \in \mathbb{N}$, are translational invariant one-body and two-body interactions, respectively, satisfying appropriate conditions (Assumption 2.1).

Throughout this paper, we set the inverse temperature $\beta(=1/kT)$ equals to 1. Then for each $\Lambda \subset \mathbb{R}^d$ the local Gibbs state is given by

$$\rho_\Lambda(A) = \mathrm{Tr}_{L^2((\mathbb{R}^d)^\Lambda)}(Ae^{-H_\Lambda})/\mathrm{Tr}_{L^2((\mathbb{R}^d)^\Lambda)}(e^{-H_\Lambda}), \qquad A \in \mathcal{A}_\Lambda \tag{1.2}$$

where $\mathcal{A}_\Lambda$ is the local algebra of bounded operators on $L^2((\mathbb{R}^d)^\Lambda)$. By the Feynman–Kac formula,[46] the density operator $\exp(-H_\Lambda)$ has its integral kernel

$$e^{-H_\Lambda}(x_\Lambda, y_\Lambda) = \int P_{x_\Lambda, y_\Lambda}(d\zeta_\Lambda)\, e^{-V(\zeta_\Lambda)} \tag{1.3}$$

where x_Λ and y_Λ are points in $(\mathbb{R}^d)^\Lambda$, $\zeta_\Lambda \in (C([0,1]; \mathbb{R}^d))^\Lambda$, $V(\zeta_\Lambda) = \int_0^1 V(\zeta_\Lambda(\tau))\, d\tau$, and P_{x_Λ, y_Λ} is the conditional Wiener measure[46] on $(C([0,1]; \mathbb{R}^d))^\Lambda$. Then the normalization factor $\mathrm{Tr}_{L^2((\mathbb{R}^d)^\Lambda)}(e^{-H_\Lambda})$ can be expressed as

$$\mathrm{Tr}_{L^2((\mathbb{R}^d)^\Lambda)}(e^{-H_\Lambda}) = \int dx_\Lambda \int P_{x_\Lambda, x_\Lambda}(d\zeta_\Lambda)\, e^{-V(\zeta_\Lambda)},$$

where dx_Λ is the Lebesgue measure on $(\mathbb{R}^d)^\Lambda$. Using the above formula we were able to define Gibbs specifications and then define the Gibbs measures on $\Omega := E^{\mathbb{Z}^\nu}$, $E := \{\omega \in C([0,1]; \mathbb{R}^d) : \omega(0) = \omega(1)\}$, in terms of

Gibbs specifications (see Section 2.1).[35] Furthermore, a characterization of Gibbs states on the quasi local C^*-algebra $\mathscr{A} := (\bigcup_{\Lambda \subset \mathbb{Z}^\nu} \mathscr{A}_\Lambda)^-$ was given through the Gibbs measures on the configuration space Ω (see ref. 35 for the details). This is so called an Euclidean approach to the problem of equilibrium quantum statistical physics of lattice systems.[1, 15–16]

As in ref. 37, we will study the Dirichlet form for the Gibbs measures in the framework of rigged Hilbert spaces.[5–8, 9, 27, 31–32, 37] For each $\gamma \in [0, 1]$, we will introduce rigged Hilbert spaces for one site spin systems and for the configuration space, respectively: For single spin systems

$$\mathsf{H}_+^{(\gamma)} \subset \mathsf{H}_0^{(\gamma)} \subset \mathsf{H}_-^{(\gamma)} \tag{1.4}$$

and for configuration space

$$\mathscr{H}_+^{(\gamma)} \subset \mathscr{H}_0^{(\gamma)} \subset \mathscr{H}_-^{(\gamma)} \tag{1.5}$$

The above embbedings are everywhere dense and belong to the Hilbert–Schmidt class. See Section 2.1 for the details. Let $\mathscr{F}_{\mathrm{loc}} C_b^\infty(\mathscr{H}_-^{(\gamma)})$ be the space of functions on $\mathscr{H}_-^{(\gamma)}$ which are smooth with bounded derivatives and depend on local and finitely many variables. For any Gibbs measure μ, we will consider the following *pre-Dirichlet form*:

$$\mathscr{E}_\mu^{(\gamma)}(u, v) := \frac{1}{2} \int \langle\!\langle \nabla u, \nabla \bar{v} \rangle\!\rangle^{(\gamma)}\, d\mu, \qquad u, v \in \mathscr{F}_{\mathrm{loc}} C_b^\infty(\mathscr{H}_-^{(\gamma)}) \tag{1.6}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle^{(\gamma)}$ is the dual pairing between $\mathscr{H}_+^{(\gamma)}$ and $\mathscr{H}_-^{(\gamma)}$ and ∇ is the gradient operator. For the form in (1.6) there corresponds a symmetric Dirichlet operator $H_\mu^{(\gamma)}$ such that the form can be represented by

$$\mathscr{E}_\mu^{(\gamma)}(u, v) = (H_\mu^{(\gamma)} u, v)_{L^2(\mu)} \tag{1.7}$$

The Dirichlet operator $H_\mu^{(\gamma)}$ is given by

$$H_\mu^{(\gamma)} u(\xi) = -\frac{1}{2} \Delta u(\xi) - \frac{1}{2} \langle\!\langle \nabla u(\xi), \beta^{(\gamma)}(\xi) \rangle\!\rangle^{(\gamma)}, \qquad \xi \in \mathscr{H}_-^{(\gamma)} \tag{1.8}$$

where Δ is the Laplacian operator and $\beta^{(\gamma)}(\xi)$ the logarithmic derivative of the measure μ.

One of the main problems is to show that for each $\gamma \in [0, 1]$, $H_\mu^{(\gamma)}$ is essentially self-adjoint with a core $\mathscr{F}_{\mathrm{loc}} C_b^\infty(\mathscr{H}_-^{(\gamma)})$. In ref. 37, we have considered the same problem for the single spin systems, i.e., for loop spaces. We have introduced a criterion for essential self-adjointness of Dirichlet operators by using an approximate method for the logarithmic derivative of a given Gibbs measure which was in turn a modification of a method originally developed by S. Albeverio, Yu. G. Kondratiev, and M. Röckner.[5–7] Our method can be applied not only to loop spaces, but also to lattice systems.

After showing the essential self-adjointness of Dirichlet operator it is worth to know some information on the potential so that the corresponding Gibbs measure is uniformly log-concave (ULC) and the Dirichlet operator $H_\mu^{(\gamma)}$ has a spectral gap at the lower end of the spectrum. We have given sufficient conditions for (ULC) which imply the uniqueness of the Gibbs measure.[9, 54] We then show that under the condition for (ULC), the unique Gibbs measure μ satisfies the log-Sobolev inequality. These properties are essential to show the L^2-ergodicity of the semi-group $T_t := \exp(-H_\mu^{(\gamma)} t)$, $t \geq 0$.

We organize this paper as follows: In Section 2.1, we discuss briefly the Gibbs measures for quantum unbounded spin systems and then introduce rigged Hilbert spaces, Dirichlet forms and associated Dirichlet operators. In Section 2.2, we state the main results of this paper. Among them, we have the essential self-adjointness of Dirichlet operators (Theorem 2.7), (ULC) of Gibbs measures (Theorem 2.10), and (LS) for Gibbs measures (Theorem 2.12). Section 3 is devoted to prove the essential self-adjointness of Dirichlet operators. In Section 4, we prove (ULC) and (LS) for Gibbs measures. In Section 5, we try to improve the results in Section 2.2 by relaxing some requirements on the interaction.

2. NOTATION, PRELIMINARIES, AND MAIN RESULTS

2.1. Notation and Preliminaries

We will use the notations introduced in refs. 32, 35, and 37. For reader's convenience, however, we will briefly review the basic notation, definitions, and preliminaries that are needed in this paper. For the details we refer to refs. 32, 35, and 37.

We denote by $\mathbb{Z}^\nu$ the ν-dimensional integer lattice space and by $\mathscr{C}$ the class of finite subsets of $\mathbb{Z}^\nu$. For $x = (x^1,..., x^d) \in \mathbb{R}^d$ and $i = (i_1,..., i_\nu) \in \mathbb{Z}^\nu$ we write

$$|x| = \left(\sum_{l=1}^{d} (x^l)^2 \right)^{1/2}, \qquad |i| = \max_{1 \leq k \leq \nu} |i_k| \tag{2.1}$$

Let dx be the Lebesgue measure on $\mathbb{R}^d$. For each $\Lambda \in \mathscr{C}$, we write

$$x_\Lambda = \{x_i : i \in \Lambda\}, \qquad dx_\Lambda = \prod_{i \in \Lambda} dx_i \tag{2.2}$$

We consider only one-body and two-body interactions. The potential function for $\Lambda \in \mathscr{C}$ is defined by

$$V(x_\Lambda) := \sum_{i \in \Lambda} \Phi_{\{i\}}(x_i) + \sum_{\{i, j\}: i, j \in \Lambda} \Phi_{\{i, j\}}(x_i, x_j) \tag{2.3}$$

where $\Phi_{\{i\}}$ and $\Phi_{\{i, j\}}$ are the interaction functions which are measurable real valued functions on $\mathbb{R}^d$ and $(\mathbb{R}^d)^2$, respectively. Throughout this paper, we impose the following conditions on the interaction:

Assumption 2.1. The interaction $\Phi = (\Phi_\Lambda)_{\Lambda \subset \mathbb{Z}^\nu}$, $|\Lambda| \leqslant 2$, satisfies the following conditions:

(a) There exist a differentiable function $P(x)$ on $\mathbb{R}^d$ and positive constants a and b such that for each $i \in \mathbb{Z}^\nu$, $\Phi_{\{i\}}(x_i) = P(x_i)$ and for some $\alpha \geqslant 2$ the bound

$$P(x) \geqslant a\,|x|^\alpha - b$$

holds. For $\alpha = 2$, the positive constant a assumes to be sufficiently large.

(b) For any positive real number $\delta > 0$, there exists a constant $M(\delta)$ such that the bound

$$|P(x)| + \sum_{l=1}^{d} \left| \frac{\partial}{\partial x^l} P(x) \right| \leqslant M(\delta) \exp(|\delta\, |x|^2)$$

holds.

(c) For each $r \in \mathbb{N}$, there exists a differentiable symmetric function $U(\cdot, \cdot; r)$: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $\Phi_{\{i, j\}}(x_i, x_j) = U(x_i, x_j; |i-j|) = U(x_j, x_i; |i-j|)$ for each $i, j \in \mathbb{Z}^\nu$. Moreover, there exists a decreasing function Ψ on $\mathbb{N}$ such that $\Psi(r) \leqslant K r^{-\nu-\varepsilon}$ for some constants K and $\varepsilon > 0$ and such that the bounds

$$|U(x, y; |i-j|)| \leqslant \Psi(|i-j|)\, \tfrac{1}{2}(|x|^2 + |y|^2)$$

hold.

(d) Let Ψ: $\mathbb{N} \to \mathbb{R}$ be the function given in the above. Then the bounds

$$\left| \frac{\partial}{\partial x^l} U(x, y; |i-j|) \right| \leqslant \Psi(|i-j|)(|x| + |y|), \qquad l = 1, 2,..., d$$

hold.

In the following we will frequently use $U(x_i, x_j)$ for $U(x_i, x_j; |i-j|)$ if there is no confusion involved.

Remark. (a) Assumption 2.1 (a) and (c) imply that the interaction is superstable and regular.[30, 35, 41] In the case for $\alpha = 2$, see ref. 26.

(b) The potential function V in (2.3) can be written as

$$V(x_\Lambda) = \sum_{i \in \Lambda} P(x_i) + \sum_{\{i, j\} \subset \Lambda} U(x_i, x_j; |i-j|) \tag{2.4}$$

for any $\Lambda \in \mathscr{C}$.

In relevance to the quantum unbounded spin systems, there corresponds a loop space $E := \mathbb{R}^d \times W_{0,0}$ to each site $i \in \mathbb{Z}^\nu$, where for $x, y \in \mathbb{R}^d$, $W_{x,y} := \{\omega \in C([0,1]; \mathbb{R}^d) : \omega(0) = x, \omega(1) = y\}$. The interval $[0, 1]$ is understood as a circle by matching the end points 0 and 1. For $\omega \in E$, we give the sup-norm $|\omega|_u$ by

$$|\omega|_u := \sup_{0 \leqslant \tau \leqslant 1} |\omega(\tau)| \tag{2.5}$$

E is equipped with the standard Borel space structure. We introduce a reference measure λ on E by

$$\lambda(d\omega) := dx\, P(d\omega), \qquad \omega \in E, \quad \omega(0) = x \tag{2.6}$$

where $P := P_{0,0}$, and $P_{x,y}$, $x, y \in \mathbb{R}^d$, is the conditional Wiener measure on $W_{x,y}$.[46] We will also write $\lambda(d\omega_\Lambda)$ instead of $\lambda^\Lambda(d\omega_\Lambda)$ for any finite $\Lambda \subset \mathbb{Z}^\nu$. The configuration space is $\Omega := E^{\mathbb{Z}^\nu}$. Let $\pi_i: \Omega \to E$ be the projection: for $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega$, $\pi_i(\xi) := \xi_i$. We topologize Ω by the countable seminorms, $\{\rho_i\}_{i \in \mathbb{Z}^\nu}$, $\rho_i(\xi) := |\pi_i(\xi)|_u$. For each subset $\Lambda \subset \mathbb{Z}^\nu$, we have a local σ-algebra $\mathscr{F}_\Lambda$, which is the minimal σ-algebra of Borel sets for which ρ_i, $i \in \Lambda$, is continuous. We simply write $\mathscr{F}$ for $\mathscr{F}_{\mathbb{Z}^\nu}$. By $\mathscr{P}(\Omega, \mathscr{F})$ we mean the space of all probability measures on $(\Omega, \mathscr{F})$.

For brevity, let us use the notation

$$\omega^2 := \int_0^1 |\omega(\tau)|^2\, d\tau, \qquad \omega \in E$$

Definition 2.2. We say that a Borel probability measure μ on $(\Omega, \mathscr{F})$ is *regular* if there exist $A^* > 0$ and $\delta > 0$ so that the projection μ_Λ

of μ on any $(\Omega, \mathscr{F}_\Lambda)$, being understood as a measure on $(E^\Lambda, \mathscr{B}(E)^\Lambda)$, satisfies

$$g(\omega_\Lambda \mid \mu) \leqslant \exp\left[-\sum_{i\in\Lambda} (A^* \omega_i^2 - \delta)\right] \tag{2.7}$$

where $g(\omega_\Lambda \mid \mu)$ is such that $\mu_\Lambda(d\omega_\Lambda) = g(\omega_\Lambda \mid \mu)\, \lambda(d\omega_\Lambda)$.

For $\Delta, \Lambda \subset \mathbb{Z}^\nu$, let us write

$$\begin{aligned} \Phi_\Delta(\zeta_\Delta) &= \int_0^1 \Phi_\Delta(\zeta_\Delta(\tau))\, d\tau \\ V(\zeta_\Lambda) &= \sum_{\Delta\subset\Lambda} \Phi_\Delta(\zeta_\Delta) \end{aligned} \tag{2.8}$$

and for $\xi \in \Omega$ and $\Lambda \subset \mathbb{Z}^\nu$,

$$W(\xi_\Lambda, \xi_{\Lambda^c}) = \sum_{i\in\Lambda,\, j\in\Lambda^c} \Phi_{\{i,j\}}(\xi_i, \xi_j) \tag{2.9}$$

Let us define

$$\begin{aligned} \mathscr{S} &:= \bigcup_{N\in\mathbb{N}} \mathscr{S}_N \\ \mathscr{S}_N &:= \left\{\xi\in\Omega: \forall l, \sum_{|i|\leqslant l} \xi_i^2 \leqslant N^2(2l+1)^\nu\right\} \end{aligned} \tag{2.10}$$

It can be shown that for any regular measure μ, $\mu(\mathscr{S}) = 1$.[35]

Definition 2.3. We say that a Borel probability measure μ on $(\Omega, \mathscr{F})$ is tempered[60] if $\mu(\mathscr{S}) = 1$.

Remark. The class of tempered probability measures on $(\Omega, \mathscr{F})$ we consider here is more restrictive than the one considered by other authors under the same terminology (see e.g., refs. 9 and 55). We will deal exclusively with the tempered Gibbs measures in the above sense. Thus, it remains open wheather the results obtained in this paper hold true or not for Gibbs measures in the more wide sense of refs. 9 and 55.

The partition function in a finite $\Lambda \subset \mathbb{Z}^\nu$ for the interaction Φ with boundary condition $\xi \in \mathscr{S}$ is defined by

$$Z_\Lambda^\Phi(\xi) := \int \lambda(d\zeta_\Lambda) \exp[-V(\zeta_\Lambda) - W(\zeta_\Lambda, \xi_{\Lambda^c})] \tag{2.11}$$

The Gibbs specification $\gamma^{\Phi} = (\gamma_{\Lambda}^{\Phi})_{\Lambda \in \mathscr{C}}$ with respect to $\mathscr{S}$ is defined by

$$\gamma_{\Lambda}^{\Phi}(A \mid \xi) := \begin{cases} Z_{\Lambda}^{\Phi}(\xi)^{-1} \int \lambda(d\zeta_{\Lambda}) \exp[-V(\zeta_{\Lambda}) - W(\zeta_{\Lambda}, \xi_{\Lambda^c})] \\ \qquad \times 1_{A}(\zeta_{\Lambda} \xi_{\Lambda^c}), \qquad \text{if} \quad \xi \in \mathscr{S} \\ 0, \qquad \text{if} \quad \xi \notin \mathscr{S} \end{cases} \tag{2.12}$$

where $A \in \mathscr{F}$ and 1_A is the indicator function of A and $\zeta_{\Lambda} \xi_{\Lambda^c}$ is the configuration coinciding with ζ_{Λ} on Λ and with ξ_{Λ^c} on Λ^c, respectively. It is easily checked that the Gibbs specification satisfies the consistent condition: for $\Delta \subset \Lambda$, $\xi \in \mathscr{S}$,

$$\gamma_{\Lambda}^{\Phi} \gamma_{\Delta}^{\Phi}(A \mid \xi) := \int_{\mathscr{S}} \gamma_{\Lambda}^{\Phi}(d\eta \mid \xi)\, \gamma_{\Delta}^{\Phi}(A \mid \eta) = \gamma_{\Lambda}^{\Phi}(A \mid \xi)$$

The Gibbs measures are now defined by

Definition 2.4. A Gibbs measure μ for the potential Φ is a tempered Borel probability measure on $(\Omega, \mathscr{F})$ satisfying the equilibrium equations

$$\mu(A) = \int \mu(d\xi)\, \gamma_{\Lambda}^{\Phi}(A \mid \xi), \qquad \Lambda \in \mathscr{C}, \quad A \in \mathscr{F}$$

We denote by $\mathscr{G}^{\Phi}(\Omega)$ the family of all Gibbs measures.

We topologize the space $\mathscr{P}(\Omega, \mathscr{F})$ with the topology of local convergence[22, 30]: for each $\mu \in \mathscr{P}(\Omega, \mathscr{F})$, the sets

$$\{\nu \in \mathscr{P}(\Omega, \mathscr{F}) : \max_{1 \leqslant k \leqslant n} |\nu(A_k) - \mu(A_k)| < \varepsilon\}$$

with $A_1, ..., A_n \in \bigcup_{\Lambda \in \mathscr{C}} \mathscr{F}_{\Lambda}$, $\varepsilon > 0$, and $n \geqslant 1$, form a base of neighborhoods of μ. In ref. 35, we have obtained the following results.

Theorem 2.5 (ref. 35, Theorem 2.7). Let the hypotheses in Assumption 2.1 (a) and (c) hold. Then any Gibbs measure is regular. Furthermore, $\mathscr{G}^{\Phi}(\Omega)$ is non-empty, convex, compact in the local convergence topology, and a Choquet simplex.

In order to study the Dirichlet forms and diffusion processes related to Gibbs measures, we introduce some rigged Hilbert spaces related to the single particle system E and also for the configuration space Ω. We begin first with the loop space.[37] Let Δ_p be the Laplacian operator on the Hilbert space $L^2[0, 1] := L^2([0, 1]; \mathbb{R}^d, d\tau)$ with periodic boundary condition and define a positive self-adjoint operator A on $L^2[0, 1]$ by

$$A := -\Delta_p + 1 \tag{2.13}$$

For each $\gamma \in [0, 1]$, let $\mathsf{H}_+^{(\gamma)}$, $\mathsf{H}_0^{(\gamma)}$, and $\mathsf{H}_-^{(\gamma)}$ be the real Hilbert spaces obtained by completions of $C^\infty([0, 1]; \mathbb{R}^d)$ with norms $|\cdot|_+^{(\gamma)}$, $|\cdot|_0^{(\gamma)}$ and $|\cdot|_-^{(\gamma)}$ induced by

$$\begin{aligned}
(\omega, \zeta)_+^{(\gamma)} &= (A\omega, A\zeta)_{L^2} \\
(\omega, \zeta)_0^{(\gamma)} &= (A^{(1-\gamma)/2}\omega, A^{(1-\gamma)/2}\zeta)_{L^2} \\
(\omega, \zeta)_-^{(\gamma)} &= (A^{-\gamma}\omega, A^{-\gamma}\zeta)_{L^2}
\end{aligned} \tag{2.14}$$

respectively. Here we have denoted by $(\cdot, \cdot)_{L^2}$ the inner product in $L^2([0, 1]; \mathbb{R}^d, d\tau)$. We note that for any $\gamma \in [0, 1]$, $A^{-(1+\gamma)/2}$ belongs to the Hilbert–Schmidt class and so

$$\mathsf{H}_+^{(\gamma)} \subset \mathsf{H}_0^{(\gamma)} \subset \mathsf{H}_-^{(\gamma)} \tag{2.15}$$

is a rigging of $\mathsf{H}_0^{(\gamma)}$ by $\mathsf{H}_+^{(\gamma)}$ and $\mathsf{H}_-^{(\gamma)}$. That is, the embeddings in (2.14) are everywhere dense and belong to the Hilbert–Schmidt class. We will use the notation $\langle \cdot, \cdot \rangle^{(\gamma)} (= (\cdot, \cdot)_0^{(\gamma)})$ for the duality between $\mathsf{H}_+^{(\gamma)}$ and $\mathsf{H}_-^{(\gamma)}$ given by the inner product in H_0. The complexification of a real Hilbert space $\mathscr{H}$ will be denoted by $\mathscr{H}_{\mathbb{C}}$.

Next, we define Hilbert spaces on the ($\mathbb{R}$-valued) sequence spaces as follows: for a given (fixed) $\sigma > 0$ put

$$\begin{aligned}
l_+ &:= \left\{ (a_i)_{i \in \mathbb{Z}^\nu} : \sum_{i \in \mathbb{Z}^\nu} e^{\sigma |i|} |a_i|^2 < \infty \right\} \\
l_0 &:= \left\{ (a_i)_{i \in \mathbb{Z}^\nu} : \sum_{i \in \mathbb{Z}^\nu} |a_i|^2 < \infty \right\} \\
l_- &:= \left\{ (a_i)_{i \in \mathbb{Z}^\nu} : \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} |a_i|^2 < \infty \right\}
\end{aligned} \tag{2.16}$$

The positive constant $\sigma > 0$ will be fixed according to Assumption 2.6 (d). Now, for each $\gamma \in [0, 1]$ the rigged Hilbert space

$$\mathscr{H}_+^{(\gamma)} \subset \mathscr{H}_0^{(\gamma)} \subset \mathscr{H}_-^{(\gamma)} \tag{2.17}$$

for the configuration space is defined by

$$\mathcal{H}_{+}^{(\gamma)} := \mathrm{H}_{+}^{(\gamma)} \otimes l_{+}, \qquad \mathcal{H}_{0}^{(\gamma)} := \mathrm{H}_{0}^{(\gamma)} \otimes l_{0}, \qquad \mathcal{H}_{-}^{(\gamma)} := \mathrm{H}_{-}^{(\gamma)} \otimes l_{-} \tag{2.18}$$

From the definitions the embeddings in $\mathcal{H}_{+}^{(\gamma)} \subset \mathcal{H}_{0}^{(\gamma)} \subset \mathcal{H}_{-}^{(\gamma)}$ are everywhere dense and belong to the Hilbert–Schmidt class. We denote the inner products and norms on $\mathcal{H}_{+}^{(\gamma)}$, $\mathcal{H}_{0}^{(\gamma)}$, and $\mathcal{H}_{-}^{(\gamma)}$ by $((\cdot,\cdot))_{+}^{(\gamma)}$, $\|\cdot\|_{+}^{(\gamma)}$, $((\cdot,\cdot))_{0}^{(\gamma)}$, $\|\cdot\|_{0}^{(\gamma)}$, and $((\cdot,\cdot))_{-}^{(\gamma)}$, $\|\cdot\|_{-}^{(\gamma)}$, respectively. Furthermore, the duality between $\mathcal{H}_{+}^{(\gamma)}$ and $\mathcal{H}_{-}^{(\gamma)}$, given by the scalar product in $\mathcal{H}_{0}^{(\gamma)}$ is denoted by $\langle\!\langle \cdot,\cdot \rangle\!\rangle^{(\gamma)}$. We see that for $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu}$

$$\begin{aligned}
(\|\xi\|_{+}^{(\gamma)})^2 &= \sum_{i \in \mathbb{Z}^\nu} e^{\sigma |i|} (|\xi_i|_{+}^{(\gamma)})^2 \\
(\|\xi\|_{0}^{(\gamma)})^2 &= \sum_{i \in \mathbb{Z}^\nu} (|\xi_i|_{0}^{(\gamma)})^2 \\
(\|\xi\|_{-}^{(\gamma)})^2 &= \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} (|\xi_i|_{-}^{(\gamma)})^2
\end{aligned} \tag{2.19}$$

For later use we also introduce some subspaces of Ω:

$$\Omega_{\log} := \{(\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega : \exists N \in \mathbb{N} \text{ s.t. } |\xi_i|_u \leqslant N \log(|i|+1), \forall i \neq 0\} \tag{2.20}$$

$$\Omega_{-} := \left\{ (\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega : \|\xi\|_u^2 := \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} |\xi_i|_u^2 < \infty \right\} \tag{2.21}$$

We see that for any $\gamma \in [0, 1]$ the inclusions $\Omega_{\log} \subset \Omega_{-} \subset \mathcal{H}_{-}^{(\gamma)}$ hold and in Lemma 2.7 of ref. 32 it was shown that for any $\mu \in \mathcal{G}^{\Phi}(\Omega)$, $\mu(\Omega_{\log}) = 1$. Therefore, we may identify $L^2(\Omega, d\mu)$ with $L^2(\mathcal{H}_{-}^{(\gamma)}, d\mu)$ (see Remark 2.9 of ref. 32).

In the following we suppress $\gamma \in [0, 1]$ in the notation. Thus $\mathcal{H}_{-}$ and $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ stand for $\mathcal{H}_{-}^{(\gamma)}$ and $\langle\!\langle \cdot,\cdot \rangle\!\rangle^{(\gamma)}$, respectively. Denote by $C^k(\mathcal{H}_{-}, B)$ the set of mappings from $\mathcal{H}_{-}$ to a Banach space B that are k-times continuously differentiable in the sense of Frechét. Define $C_b^k(\mathcal{H}_{-}, B)$ as the subset of $C^k(\mathcal{H}_{-}, B)$ which is characterized by the boundedness in usual operator norms of the derivatives

$$f^{(l)} : \mathcal{H}_{-} \to \mathcal{L}(\mathcal{H}_{-}, \mathcal{L}(\mathcal{H}_{-},\ldots, \mathcal{L}(\mathcal{H}_{-}, B)\ldots)), \qquad l = 0, 1,\ldots, k$$

For $f: \mathcal{H}_{-} \to \mathbb{C}$, identify $f'(\cdot) \in \mathcal{L}(\mathcal{H}_{-}, \mathbb{C})$ with the vector $\hat{f}'(\cdot) \in \mathcal{H}_{+,\mathbb{C}}$ and $f''(\cdot)$ with the operator $\hat{f}''(\cdot) \in \mathcal{L}(\mathcal{H}_{-}, \mathcal{H}_{+,\mathbb{C}})$ by the formulas

$$\begin{aligned}
f'(\xi)\,\zeta &= \langle\!\langle \hat{f}'(\xi), \zeta \rangle\!\rangle \\
(f''(\xi)\,\zeta)\,\phi &= \langle\!\langle \hat{f}''(\xi)\,\zeta, \phi \rangle\!\rangle, \qquad \zeta, \phi, \xi \in \mathcal{H}_{-}
\end{aligned} \tag{2.22}$$

For the function $f \in C_b^2 \equiv C_b^2(\mathscr{H}_-, \mathbb{C})$ we use the symbol $\nabla f = f'$ and $\Delta f = \mathrm{Tr}_{\mathscr{H}_0}(f'')$. We introduce in the space $C_b^2(\mathscr{H}_-)$ the norm

$$\|f\|_{C_b^2(\mathscr{H}_-)} := \sup_{\xi \in \mathscr{H}_-} \{ |f(\xi)| + \|f'(\xi)\|_+ + \|f''(\xi)\|_{\mathscr{L}(\mathscr{H}_-, \mathscr{H}_{+,\mathbb{C}})} \}$$

We introduce also the set $C^k_{\mathrm{pol}}(\mathscr{H}_-, B) \subset C^k(\mathscr{H}_-, B)$ of all "polynomially bounded" mappings,[7] i.e., which satisfy

$$\|f^{(l)}(\xi)\| \leqslant C(1 + \|\xi\|_-)^p, \qquad \xi \in \mathscr{H}_-$$

for some $p \in \mathbb{N}$ in the corresponding operator norms of the derivatives $f^{(l)}$, $l = 0,..., k$. For example, for any $f \in C^2_{\mathrm{pol}}(\mathscr{H}_-) := C^2_{\mathrm{pol}}(\mathscr{H}_-, \mathbb{C})$ there exist $C > 0$, $p \in \mathbb{N}$ such that

$$|f(\xi)| + \|f'(\xi)\|_+ + \|f''(\xi)\|_{\mathscr{L}(\mathscr{H}_-, \mathscr{H}_{+,\mathbb{C}})} \leqslant C(1 + \|\xi\|_-)^p, \qquad \xi \in \mathscr{H}_- \quad (2.23)$$

Finally let $\mathscr{K} \subset \mathscr{H}_+$ be a dense linear subset in the Hilbert space $\mathscr{H}_+$. We denote by $\mathscr{F}_{\mathscr{K}} C_b^k(\mathscr{H}_-)$, $k = 0,..., \infty$, the set of all $\mathscr{K}$-cylinder functions on $\mathscr{H}_-$ with all derivatives up to k bounded, e.g., $\mathscr{F}_{\mathscr{K}} C_b^\infty(\mathscr{H}_-)$ is the set of all functions on $\mathscr{H}_-$ such that there exist $N \subset \mathbb{N}$, $\{\phi_1,..., \phi_N\} \subset \mathscr{K}$ and $f_N \in C_b^\infty(\mathbb{R}^N)$ such that

$$f(\xi) = f_N(\langle\langle \xi, \phi_1 \rangle\rangle,..., \langle\langle \xi, \phi_N \rangle\rangle), \qquad \xi \in \mathscr{H}_-$$

We would like to take a special subset $\mathscr{K}_0 \subset \mathscr{H}_+$. Let $\mathscr{P}_{\mathrm{fin}}$ be the space of finite rank projections P on H_0. P extends continuously to H_- and we use the same notation P for the extention. Let $\{P_n\}_{n \in \mathbb{N}} \subset \mathscr{P}_{\mathrm{fin}}$ be a fixed increasing sequence such that $\bigcup_{n \in \mathbb{N}} P_n$ is dense in H_0. We define

$$\begin{aligned} \mathscr{F}_{\mathrm{loc}} C_b^k &:= \mathscr{F}_{\mathscr{K}_0} C_b^k \\ \mathscr{K}_0 &:= \mathrm{span}\{\phi \in \mathscr{H}_+ : \exists \Delta \in \mathscr{C} \text{ and } n \in \mathbb{N} \text{ s.t. } \pi_i \phi = 0 \text{ if } i \notin \Delta \\ &\qquad \text{and } \pi_i \phi \in P_n \ \forall i \in \Delta\} \end{aligned} \quad (2.24)$$

For given $\gamma \in [0, 1]$ and $\mu \in \mathscr{G}^{\Phi}(\Omega)$, we define the *pre-Dirichlet form* by

$$\begin{aligned} D(\mathscr{E}_\mu^{(\gamma)}) &:= \mathscr{F}_{\mathrm{loc}} C_b^\infty(\mathscr{H}_-^{(\gamma)}) \\ \mathscr{E}_\mu^{(\gamma)}(u, v) &:= \tfrac{1}{2} \int_{\mathscr{H}_-^{(\gamma)}} \langle\langle \nabla u, \nabla \bar{v} \rangle\rangle^{(\gamma)} \, d\mu, \qquad u, v \in D(\mathscr{E}_\mu^{(\gamma)}) \end{aligned} \quad (2.25)$$

In ref. 32, we have shown that the form $\mathscr{E}_\mu^{(0)}$ for $\gamma = 0$ is closable and the closure is a Dirichlet form. Furthermore, there exists a diffusion process which is properly associated with $\mathscr{E}_\mu^{(0)}$. By the same method we can show the same results for the form $\mathscr{E}_\mu^{(\gamma)}$ for any $\gamma \in [0, 1]$. In ref. 32, we have also shown that the form $\mathscr{E}_\mu^{(0)}$, and hence the form $\mathscr{E}_\mu^{(\gamma)}$ for any $\gamma \in [0, 1]$ too, is related by the Dirichlet operator defined in the following way. Let $P(x)$ and $U(x_i, x_j; |i-j|)$ be the one-body and two-body interactions given in Assumption 2.1. Let us define

$$\tilde{P}(x) := P(x) - \tfrac{1}{2}|x|^2, \qquad x \in \mathbb{R}^d \tag{2.26}$$

For $\xi \in \Omega_{\log}$, let $\beta^{(\gamma)}(\xi) \in \mathscr{H}_-$ be defined by

$$\begin{aligned} &\beta^{(\gamma)}(\xi) := (\beta_i^{(\gamma)}(\xi))_{i \in \mathbb{Z}^\nu} \\ &\beta_i^{(\gamma)}(\xi) := -\left[A^\gamma \xi_i + A^{-(1-\gamma)} \left\{ \partial\tilde{P}(\xi_i) + \sum_{j \in \mathbb{Z}^\nu : j \neq i} \partial^i U(\xi_i, \xi_j; |i-j|) \right\} \right] \end{aligned} \tag{2.27}$$

where $\partial\tilde{P}$ is the gradient of $\tilde{P}$ on $\mathbb{R}^d$ and $\partial^i U(x_i, x_j; |i-j|)$ is the gradient of U with respect to the x_i variable. For $\xi_i \in E$, $\partial\tilde{P}(\xi_i) \in E$ is defined by $\partial\tilde{P}(\xi_i)(\tau) := \partial\tilde{P}(\xi_i(\tau))$, $\tau \in [0, 1]$.

For $\gamma \in [0, 1]$, let us define the Dirichlet operators $H_\mu^{(\gamma)}$ by

$$\begin{aligned} &D(H_\mu^{(\gamma)}) := C_b^2(\mathscr{H}_-^{(\gamma)}) \\ &H_\mu^{(\gamma)} u(\xi) := -\tfrac{1}{2}\Delta u(\xi) - \tfrac{1}{2}\langle\!\langle \nabla u(\xi), \beta^{(\gamma)}(\xi) \rangle\!\rangle^{(\gamma)}, \qquad u \in D(H_\mu^{(\gamma)}) \end{aligned} \tag{2.28}$$

By Lemma 3.2(b) in Section 3, for each $\gamma \in [0, 1]$ the Dirichlet operator $H_\mu^{(\gamma)}$ is well-defined on $C_b^2(\mathscr{H}_-^{(\gamma)})$. It can be shown that the form in (2.25) is related by the Dirichlet operator in (2.28) in the following way:

$$\mathscr{E}_\mu(u, v) = (H_\mu u, v)_{L^2(\mu)}, \qquad u, v \in D(\mathscr{E}_\mu) \tag{2.29}$$

See ref. 32 for the details.

2.2. Main Results

The main purposes of this paper are to show the essential self-adjointness of Dirichlet operators in (2.28) with minimum definition of domains and to show some related results. For these purposes we need some additional assumptions on the interaction functions.

Assumption 2.6. Let $P(x)$ and $U(x, y; |i-j|)$ be the one-body and two-body interactions introduced in Assumption 2.1. We assume further that the following properties hold: $P(x)$ and $U(x, y; |i-j|)$, $i, j \in \mathbb{Z}^\nu$, are three times continuously differentiable functions satisfying the following conditions:

(a) For any positive real number $\delta > 0$, there exists positive constant $M(\delta)$ such that the bound

$$\sum_{l,k=1}^{d} \left| \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^k} P(x) \right| \leqslant M(\delta) \exp(\delta |x|^2)$$

holds.

(b) There exists $M \in \mathbb{R}$ such that

$$\text{Hess.}\, P(x) \geqslant M\mathbf{1}, \qquad x \in \mathbb{R}^d$$

where Hess. $P(x)$ is the Hessian of $P(x)$, i.e., the $d \times d$ matrix whose $l-k$ elements are given by $((\partial/\partial x^l)(\partial/\partial x^k)\, P(x))$, $l, k = 1, 2,..., d$.

(c) In the case of $d \geqslant 2$, there exist a function $Q: \mathbb{R} \to \mathbb{R}$ and an element $b \in \mathbb{R}^d$ such that $P(x) = Q(|x|) + b \cdot x$, $x \in \mathbb{R}^d$.

(d) The function Ψ of Assumption 2.1 (b) is exponentially decreasing: there exist $K > 0$ and $\sigma > 0$ such that

$$\Psi(r) \leqslant K e^{-2\sigma r}, \qquad r \in \mathbb{N}$$

Furthermore, the bounds

$$\left| \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^k} U(x, y; |i-j|) \right| + \left| \frac{\partial}{\partial x^l} \frac{\partial}{\partial y^k} U(x, y; |i-j|) \right| \leqslant \Psi(|i-j|), \quad l, k = 1, 2,..., d$$

hold and for any $y \in \mathbb{R}^d$ and $r \in \mathbb{N}$ the third order partial derivatives of $U(x, y; r)$ with respect to x and y variables assume to be bounded by $\Psi(r)$.

It would be worth to give comments on Assumption 2.6(c)–(d). Assumption 2.6(c) for $d \geqslant 2$ is introduced for technical reasons. We believe that the restriction is unnecessary. In Section 5, we give a possible relaxation of Assumption 2.6(c). See Theorem 5.1. If the one – body interaction P and its derivatives up to third order are polynomially bounded, i.e., $P \in C^{(3)}_{\text{pol}}(\mathbb{R}^d, \mathbb{R})$, then Assumption 2.6(d) can be relaxed. See Theorem 5.2. We now state the first main result of this paper:

Theorem 2.7. (a) Suppose that the hypotheses in Assumption 2.1 and Assumption 2.6 hold and $\mu \in \mathcal{G}^{\Phi}(\Omega)$. Then, for each $\gamma \in [0, 1]$ the Dirichlet operator $H^{(\gamma)}_{\mu}$ is essentially self-adjoint on the domain $C^2_b(\mathcal{H}^{(\gamma)}_{-})$ in $L^2(\mathcal{H}^{(\gamma)}_{-}, d\mu)$.

(b) Under the same hypotheses as in the above, $\mathcal{F}_{\text{loc}} C^{\infty}_b(\mathcal{H}^{(\gamma)}_{-})$ is also a domain of essential self-adjointness of $H^{(\gamma)}_{\mu}$ for any $\gamma \in [0, 1]$.

The proof of the theorem will be given in the next section. From now on, we discuss the log-concavity of Gibbs measures and related results. For $\gamma \in [0, 1]$, let us define an operator $R_\mu^{(\gamma)}(\xi)\colon \mathscr{H}_+^{(\gamma)} \to \mathscr{H}_-^{(\gamma)}$ by

$$R_\mu^{(\gamma)}(\xi)(\phi) := -\frac{\partial \beta^{(\gamma)}(\xi)}{\partial \phi}, \qquad \phi \in \mathscr{H}_+ \tag{2.30}$$

We will show in Lemma 4.1 that $\forall \xi \in \Omega_{\log}$, $R_\mu^{(\gamma)}(\xi) \in \mathscr{L}(\mathscr{H}_+^{(\gamma)}, \mathscr{H}_-^{(\gamma)})$.

Definition 2.8. We say that a Gibbs measure μ is uniformly log-concave (ULC) in the form $\mathscr{E}_\mu^{(\gamma)}$ or $R_\mu^{(\gamma)}$-positive if there exists $\lambda > 0$ such that for any $\phi \in \mathscr{H}_+^{(\gamma)}$ and μ-a.a. $\xi \in \mathscr{H}_-^{(\gamma)}$, the bound

$$\langle\!\langle R_\mu(\xi)\,\phi, \phi \rangle\!\rangle^{(\gamma)} \geqslant \lambda(\|\phi\|_0^{(\gamma)})^2$$

holds.

In the following we again suppress $\gamma \in [0, 1]$ in the notation if there is no confusion involved. Let us fix a dense linear subset $\mathscr{K} \subset \mathscr{H}_+$. We say that a measure $\mu \in \mathscr{P}(\mathscr{H}_-)$ is $\mathscr{K}$-ergodic iff the only measurable subsets of $\mathscr{H}_-$ which are $\mathscr{K}$-invariant have μ-measure zero or one. We recall that a μ-measurable set $A \subset \mathscr{H}_-$ is $\mathscr{K}$-invariant if $\forall \phi \in \mathscr{K}$, $\mu((A \backslash A_\phi) \cup (A_\phi \backslash A)) = 0$, where $A_\phi = A + \phi = \{\xi + \phi : \xi \in A\}$.

We define the space $W_2^1(\mu)$ as the closure of $C_b^2(\mathscr{H}_-)$ with respect to the norm

$$\|u\|_{W_2^1(\mu)}^2 := \int_{\mathscr{H}} \{|u(\xi)|^2 + \|u'(\xi)\|_+^2\}\, d\mu(\xi)$$

Correspondingly, the space $W_2^2(\mu)$ is the closure of $C_b^2(\mathscr{H}_-)$ in the norm

$$\|u\|_{W_2^2(\mu)}^2 := \|u\|_{W_2^1(\mu)}^2 + \int_{\mathscr{H}} \operatorname{Tr}_{\mathscr{H}_0}(u''(\xi)\,\bar{u}''(\xi))\, d\mu(\xi)$$

As in ref. 7, we denote by $\mathscr{P}_{\text{sa}}(\mathscr{H}_-) \subset \mathscr{P}(\mathscr{H}_-)$ the set of all probability measures in $\mathscr{H}_-$ which is characterized by the following two conditions:

(a) For any $\mu \in \mathscr{P}_{\text{sa}}(\mathscr{H}_-)$ there exists the square integrable logarithmic derivative β_μ of μ and therefore the Dirichlet operator H_μ is well defined on $C_b^2(\mathscr{H}_-)$ by the formula (2.28).

(b) H_μ is essentially self-adjoint in $L^2(\mathscr{H}_-, d\mu)$ with a core $C_b^2(\mathscr{H}_-)$.

From now on, if $\mu \in \mathscr{P}_{\text{sa}}(\mathscr{H}_-)$, for simplicity, we write the same notation H_μ for the closure of H_μ. The following theorem was proven in ref. 7.

Theorem 2.9 (ref. 7, Theorem 2). Suppose $\mu \in \mathscr{P}_{sa}(\mathscr{H}_-)$ is (ULC). Then,

(a) $D(H_\mu) \supset W_2^2(\mu)$.

(b) If the measure μ is $\mathscr{K}$-ergodic, then the point $0 \in \mathbb{R}$ is a simple eigenvalue of H_μ.

(c) If the measure μ is $\mathscr{K}$-ergodic, then there is a gap in the spectrum of the operator H_μ; moreover, $H_\mu \geqslant \frac{1}{4}\lambda$ on the orthogonal complement to the constants in $L^2(\mathscr{H}_-, d\mu)$.

For the (ULC) of Gibbs measures we have the following result.

Theorem 2.10. Suppose the hypotheses in Assumption 2.1 and Assumption 2.6 hold. In addition, suppose the hypotheses of Assumption 2.6 (b) holds with a positive constant $M > 0$, i.e., $\exists M > 0$ such that

$$\text{Hess.}\, P(x) \geqslant M\mathbf{1}, \qquad x \in \mathbb{R}^d$$

Furthermore, suppose that

$$M' := 2d \sum_{i \in \mathbb{Z}^\nu:\, i \neq 0} \Psi(|i|) < M$$

Then, for each $\gamma \in [0, 1]$, any $\mu \in \mathscr{G}^\Phi(\Omega)$ is (ULC) in the form $\mathscr{E}_\mu^{(\gamma)}$ with a concavity constant $\lambda := \min\{M - M', (M - M')^\gamma\} > 0$.

The proof of the theorem will be given in Section 4. Let us take the subspace $\mathscr{K} \subset \mathscr{H}_+$ in Theorem 2.9 to be the special one $\mathscr{K}_0$ defined in (2.24). We say that a Dirichlet form $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ is irreducible if for any $u \in D(\mathscr{E}_\mu)$ with $\mathscr{E}_\mu(u, u) = 0$ it follows that u is constant μ-a.e. In ref. 8, it was shown that irreducibility of $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ is equivalent with the extremality of μ on the set of measures that have the same logarithmic derivatives. By Theorem 3.4 and Theorem 3.7 of ref. 8, the irreducibility in turn is equivalent with (space) ergodicity of μ under the condition of equation (3.2) of ref. 8. For Gibbs measures, by the Gibbs property, the condition (3.2) of ref. 8 holds true for any $k \in \mathscr{K}_0$ through equation (3.3) of ref. 8. On the other hand, under the condition of (ULC) in Theorem 2.10, a direct application of the method used in refs. 9 and 54 shows that the Gibbs measure exists uniquely. Thus, the unique Gibbs measure is automatically an extremal one. Using this remark and Theorem 2.9, we state as a corollary the following result.

Corollary 2.11. Suppose the hypotheses in Theorem 2.10 hold and let $\mu \in \mathscr{G}^\Phi(\Omega)$ be the unique Gibbs measure. Then, the conclusions (a), (b), and (c) of Theorem 2.9 hold for the Dirichlet operator $H_\mu^{(\gamma)}$ for any $\gamma \in [0, 1]$.

From now on, we discuss the log-Sobolev inequality for Gibbs measures. Let us recall that a probability measure μ satisfies a log-Sobolev inequality (LS) (in the form $\mathscr{E}_\mu^{(\gamma)}$) if and only if there exists some $c_\mu > 0$ such that for all $f \in W_2^1$ the following inequality holds[23]:

$$\int_{\mathscr{H}_-} |f(\xi)|^2 \log |f(\xi)| \, d\mu(\xi)$$
$$\leqslant c_\mu \int_{\mathscr{H}_-} (\|\nabla^{(\gamma)} f(\xi)\|_0^{(\gamma)})^2 \, d\mu(\xi) + \|f\|^2_{L^2(\mu)} \log \|f\|_{L^2(\mu)} \tag{2.31}$$

The coefficient c_μ is called the *Sobolev coefficient*. An important consequence of the log-Sobolev inequality is that the semi-group $T_t := e^{-tH_\mu}$, $t \geqslant 0$, is hypercontractive.[23]

We have the following result on (LS).

Theorem 2.12. Suppose the hypotheses in Theorem 2.10 hold. Then, the unique Gibbs measure $\mu \in \mathscr{G}^\Phi(\Omega)$ satisfies the log-Sobolev inequality in the form $\mathscr{E}_\mu^{(\gamma)}$ for any $\gamma \in [0, 1]$ with a Sobolev coefficient $c_\mu = \tilde{\lambda}^{-1}$, where $\tilde{\lambda} = \min\{1, M - M'\}$.

The proof of the theorem will be given in Section 4.

Let us define a semi-group $(T_t)_{t \geqslant 0}$ in the space $L^2(\mu)$ by

$$T_t := \exp(-tH_\mu^{(\gamma)}) \tag{2.32}$$

From Theorem 2.12 and Rothaus–Simon mass gap theorem,[40, 45] we have that $0 \in \mathbb{R}$ is a simple eigenvalue for H_μ and $H_\mu \geqslant (2c_\mu)^{-1}$ on the orthogonal complement to the constants in $L^2(\mu)$. By the spectral theorem, this implies the L^2-ergodicity of the semi-group T_t, $t \geqslant 0$:

$$\forall f \in L^2(\mu), \forall t \geqslant 0, \|T_t f - E_\mu f\|_{L^2(\mu)} \leqslant \exp\left(-\frac{t}{2c_\mu} \|f - E_\mu f\|_{L^2(\mu)}\right) \tag{2.33}$$

where $E_\mu f = \int_{\mathscr{H}_-} f(\xi) \, d\mu(\xi)$. It is obvious that μ is invariant under the action of T_t, $t \geqslant 0$.

We will briefly discuss the Markov process associated with the semi-group $(T_t)_{t \geqslant 0}$ in Section 5.

3. ESSENTIAL SELF-ADJOINTNESS OF DIRICHLET OPERATOR

In this section we prove the essential self-adjointness of the Dirichlet operator $H_\mu^{(\gamma)}$ (Theorem 2.7). We shall use an approximate criterion for essential self-adjointness of Dirichlet operators associated with Dirichlet forms given by probability measures on Hilbert spaces. The approximate criterion we will use is a modified version of the criterion given by Albeverio, Kondrative and Röckner (ref. 7, Theorem 1).

We begin with the general formalism of Dirichlet forms and Dirichlet operators in the framework of rigged Hilbert spaces. Let $\mathscr{H}_0$ be a separable Hilbert space with the scalar product $(\cdot,\cdot)_0$ and norm $|\cdot|_0$ and let

$$\mathscr{H}_+ \subset \mathscr{H}_0 \subset \mathscr{H}_-$$

be a rigging of $\mathscr{H}_0$ by the Hilbert spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ with scalar products and norms $(\cdot,\cdot)_+$, $|\cdot|_+$ resp. $(\cdot,\cdot)_-$, $|\cdot|_-$. We suppose the embeddings in the above are everywhere dense and belong to the Hilbert–Schmidt class.[7] We also suppose that $|\cdot|_- \leqslant |\cdot|_0 \leqslant |\cdot|_+$. Otherwise it is sufficient to renorm $\mathscr{H}_+$. The duality between $\mathscr{H}_+$ and $\mathscr{H}_-$ is given by the scalar product in $\mathscr{H}_0$ and will be denoted by $<\cdot,\cdot>$.

Let μ be a probability measure on the Borel σ-algebra $\mathscr{B}(\mathscr{H}_-)$ which is quasi-invariant under translations by the vectors in a dense subset of $\mathscr{H}_+$. Let β: $\mathscr{H}_- \to \mathscr{H}_-$ be the logarithmic derivative of the measure μ. See section 2 and refs. 5–7. We assume that for any $p \in \mathbb{N}$

$$\int_{\mathscr{H}_-} |x|_-^p \, d\mu(x) < \infty \tag{3.1}$$

and

$$\int_{\mathscr{H}_-} |\beta(x)|_-^2 \, |x|_-^p \, d\mu(x) < \infty \tag{3.2}$$

Let H_μ be the Dirichlet operator on the domain $D(H_\mu) = C^2_{\text{pol}}(\mathscr{H}_-)$ given by the formula.

$$(H_\mu u)(x) = -\tfrac{1}{2}\Delta u(x) - \tfrac{1}{2}\langle \nabla u(x), \beta(x)\rangle, \qquad u \in C^2_{\text{pol}}(\mathscr{H}_-), \quad x \in \mathscr{H}_- \tag{3.3}$$

Let $\mathscr{K}_+$ and $\mathscr{K}_-$ be real separable Hilbert spaces with inner products and norms $\langle\cdot,\cdot\rangle_{\mathscr{K}_+}$, $|\cdot|_{\mathscr{K}_+}$ resp. $\langle\cdot,\cdot\rangle_{\mathscr{K}_-}$, $|\cdot|_{\mathscr{K}_-}$. Suppose that the inclusions

$$\mathscr{H}_+ \subset \mathscr{K}_+ \subset \mathscr{H}_0 \subset \mathscr{K}_- \subset \mathscr{H}_- \tag{3.4}$$

holds, and suppose that the duality between $\mathscr{H}_+$ and $\mathscr{H}_-$ is given by the inner product of $\mathscr{H}_0$ and will be denoted by $\langle \cdot, \cdot \rangle$. However, we do not assume that the embeddings $\mathscr{H}_+ \subset \mathscr{H}_0 \subset \mathscr{H}_-$ belong to the Hilbert-Schmidt class.

The following is a result of ref. 37 which is a modified version of ref. 7, Theorem 1. See also ref. 37, Theorem 3.1.

Theorem 3.1 (ref. 37, Theorem 3.2). Let μ be a probability measure on $\mathscr{B}(\mathscr{H}_-)$ which satisfies the conditions (3.1) and (3.2). Let β can be written as $\beta = \beta_1 + \beta_2$. Suppose that there exists a sequence $\{b_n : n \in \mathbb{N}\}$, $b_n : \mathscr{H}_- \to \mathscr{H}_-$, $b_n = b_{1,n} + b_{2,n}$, $n \in \mathbb{N}$, such that the following properties hold:

(i) For each $n \in \mathbb{N}$, $b_n \in C^2_{\text{pol}}(\mathscr{H}_-, \mathscr{H}_-)$.

(ii) For each $n \in \mathbb{N}$, there exists a constant $c(n) \geqslant 0$ such that

$$\langle b_n(x), x \rangle_- \leqslant c(n)(1 + |x|^2_-), \qquad x \in \mathscr{H}_-$$

(iii) For any $n \in \mathbb{N}$, there exists a contant $M(n) \geqslant 0$ such that the bound

$$\|b_n'(x)\|_{\mathscr{L}(\mathscr{H}_-, \mathscr{H}_-)} \leqslant M(n)$$

holds uniformly in $x \in \mathscr{H}_-$.

(iv) For any $n \in \mathbb{N}$, there exists a constant $c_1(n)$ such that for any $h \in \mathscr{H}_-$, the bound

$$\langle h, b_n'(x)\, h \rangle_- \leqslant c_1(n)\, |h|^2_-$$

holds uniformly in $x \in \mathscr{H}_-$.

(v) There exists a constant $c_2 \geqslant 0$ and $N_0 \in \mathbb{N}$ such that for any $n \geqslant N_0$ and $h \in \mathscr{H}_-$ the bound

$$\langle h, b_n'(x)\, h \rangle_{\mathscr{H}_-} \leqslant c_2\, |h|^2_{\mathscr{H}_-}$$

holds uniformly in $x \in \mathscr{H}_-$.

(vi) There exists a sequence $\{a_n : n \in \mathbb{N}\}$ of positive real numbers such that for the constants $c_1(n)$, $n \in \mathbb{N}$, appeared in (iv)

$$a_n \exp(c_1(n)/2) \to 0 \qquad \text{as} \quad n \to \infty$$

and such that for any $n \in \mathbb{N}$

$$\| |\beta_{1,n} - b_{1,n}|_- \|_{L^2(\mu)} \leqslant a_n$$

(vii) $\| |\beta_2 - b_{2,n}|_{\mathscr{H}_-} \| \to 0$ as $n \to \infty$.

Then the Dirichlet operator H_μ is essentially self-adjoint on $C_b^2(\mathscr{H}_-)$.

For the reader's convenience, we describe the main ideas of the proof of Theorem 3.1. For the complete proof, we refer to ref. 37. For any $n \in \mathbb{N}$ we define a differential operator H_n on the domain $C^2_{\text{pol}}(\mathscr{H}_-)$ by the formula

$$H_n u = -\tfrac{1}{2}\Delta u - \tfrac{1}{2}\langle \nabla u, b_n \rangle \tag{3.5}$$

We have used the following general parabolic criterion of essential self-adjointness (see ref. 18, Chapter 5, Theorem 1.10): Let us consider the Cauchy problem

$$\begin{aligned} &\frac{d}{dt} u_n(t) + H_n u_n(t) = 0 \\ &u_n(0) = f, \qquad t \in [0, 1] \end{aligned} \tag{3.6}$$

where $f \in C^2_{\text{pol}}(\mathscr{H}_-)$ is arbitrary. If one can show the existence of strong solutions

$$u_n : [0, 1] \to L^2(\mu)$$

for (3.6) such that

$$u_n(t) \in D(H_n) \text{ for any } t \in [0, 1] \text{ and } n \in \mathbb{N} \tag{3.7}$$

and

$$\int_0^1 \|(H_\mu - H_n)\, u_n(t)\|_{L^2(\mu)}\, dt \to 0, \qquad n \to \infty \tag{3.8}$$

Then H_μ is essentially self-adjoint on $C_b^2(\mathscr{H}_-)$.

The existence and the differentiability of the strong solutions of (3.6) satisfying (3.7) are guaranteed by the conditions (i) and (ii) in Theorem 3.1. See ref. 7 and referneces therein. Thus one only needs to show (3.8). It follows from (3.3) and (3.5) that

$$\int_0^1 \|(H_\mu - H_n)\, u_n(t)\|_{L^2(\mu)}\, dt$$
$$\leqslant \int_0^1 \|\langle \beta_1 - b_{1,n}, \nabla u_n(t)\rangle\|_{L^2(\mu)}\, dt + \int_0^1 \|\langle \beta_2 - b_{2,n}, \nabla u_n(t)\rangle\|_{L^2(\mu)}\, dt$$
$$\leqslant \int_0^1 \|(|\beta_1 - b_{1,n}|_-)(|\nabla u_n(t)|_+)\|_{L^2(\mu)}\, dt$$
$$+ \int_0^1 \|(|\beta_2 - b_{2,n}|_{\mathscr{K}_-})(|\nabla u_n(t)|_{\mathscr{K}_+})\|_{L^2(\mu)}\, dt \tag{3.9}$$

Using a method of stochastic analysis[7, 37] and the conditions (iii)–(v), it can be shown that

$$|\nabla u_n(t)|_+ \leqslant \|f\|_{C_b^2} \exp(c_1(n)\, t/2) \tag{3.10}$$

and

$$|\nabla u_n(t)|_{\mathscr{K}_+} \leqslant \|f\|_{C_b^2} \exp(c_2 t/2) \tag{3.11}$$

The proofs of the inequalities in (3.10) and (3.11) are given in ref. 37 (Lemma 3.1 (a) and the bound (3.18) in ref. 37, respectively) in full details. For the sake of self-containedness, we give a short-hand version of the proof of (3.11) below.

The solution u_n of (3.6) can be found to be

$$u_n(t, x) = E\{f(\xi_{n,x}(t))\}, \qquad x \in \mathscr{H}_-, \quad t \in \mathbb{R}_+ \tag{3.12}$$

where $\xi_{n,x}(t)$ satisfies the stochastic differential equation:

$$d\xi_{n,x}(t) = \tfrac{1}{2} b_n(\xi_{n,x}(t))\, dt + dw(t)$$
$$\xi_{n,x}(0) = x \in \mathscr{H}_-$$

Here, w: $[0, \infty) \to \mathscr{H}_-$ is a standard Wiener process which corresponds to the Hilbert space $\mathscr{H}_0$. The equation has a unique solution $\xi_{n,x}$ and it has various differentiable properties (see ref. 7). Let $\eta_h(t)$ be the directional derivative of $\xi_{n,x}(t)$ in the direction h:

$$\eta_h(t) := \xi'_{n,x}(t)\, h \tag{3.13}$$

Then we have

$$\eta_h(t) = h + \tfrac{1}{2} \int_0^t b'_n(\xi_{n,x}(\tau))\, \eta_h(\tau)\, d\tau \tag{3.14}$$

We first show that $\eta_h(t) \in \mathscr{H}_-$ for any $h \in \mathscr{H}_-$. It follows from (3.14) and condition (iii) of Theorem 3.1 that for any $h \in \mathscr{H}_-$

$$|\eta_h(t)|_{\mathscr{H}} \leqslant |h|_{\mathscr{H}} + \tfrac{1}{2}M(n)\int_0^t |\eta_h(\tau)|_-\,d\tau, \qquad P\text{-a.s.} \tag{3.15}$$

Using the expression for $\eta_h(t)$ and the condition (iv) we have

$$|\eta_h(\tau)|_- \leqslant |h|_- \exp(c_1(n)\,\tau/2), \qquad P\text{-a.s.} \tag{3.16}$$

Substituting (3.16) into (3.15) and using the fact that $|\cdot|_- \leqslant |\cdot|_{\mathscr{H}}$, we obtain that $|\eta_h(t)|_{\mathscr{H}_-} < \infty$. This proves that $\eta_h(t) \in \mathscr{H}_-$, P-a.s. It follows from (3.12) that for any $x \in \mathscr{H}_-$ and $h \in \mathscr{H}_-$

$$\begin{aligned}\langle \nabla u_n(t,x), h\rangle &= u_n'(t,x)\,h \\ &= E\{\langle f'(\xi_{n,x}(t)), \eta_h(t)\rangle\} \\ &\leqslant \{\sup_{x\in\mathscr{H}} |f'(x)|_{\mathscr{H}_+}\}\, E(|\eta_h(t)|_{\mathscr{H}}) \\ &\leqslant \|f\|_{C_b^2}\, E(|\eta_h(t)|_{\mathscr{H}})\end{aligned} \tag{3.17}$$

On the other hand, we obtain from condition (v) that

$$\begin{aligned}|\eta_h(t)|^2_{\mathscr{H}_-} - |h|^2_{\mathscr{H}_-} &= \int_0^t \frac{d}{d\tau}|\eta_h(\tau)|^2_{\mathscr{H}}\,d\tau \\ &= \int_0^t \langle \eta_h(\tau), b_n'(\xi_{n,x}(\tau))\,\eta_h(\tau)\rangle\,d\tau \\ &\leqslant c_2\int_0^t |\eta_h(\tau)|^2_{\mathscr{H}}\,d\tau\end{aligned}$$

By the Gronwall's inequality we obtain

$$|\eta_h(t)|^2_{\mathscr{H}} \leqslant |h|^2_{\mathscr{H}}\, e^{c_2 t} \tag{3.18}$$

Since

$$|\nabla u_n(t,x)|_{\mathscr{H}_+} = \sup_{h\in\mathscr{H}\,:\,|h|_{\mathscr{H}_-}=1} |\langle \nabla u_n(t,x), h\rangle|$$

the inequality (3.11) follows from (3.17) and (3.18).

Now (3.8) follows from (3.9)–(3.11), the conditions (vi) and (vii) in Theorem 3.1. For the details, we refer the reader to ref. 37.

We shall use Theorem 3.1 to prove Theorem 2.7 (a). Recall the definitions of l_+, l_0 and l_- in (2.16). For each $\gamma \in [0, 1]$, let $\mathscr{K}_+^{(\gamma)}$ and $\mathscr{K}_-^{(\gamma)}$ be the Hilbert spaces defined by

$$\mathscr{K}_+^{(\gamma)} = \mathsf{H}_0^{(\gamma)} \otimes l_+, \qquad \mathscr{K}_-^{(\gamma)} = \mathsf{H}_0^{(\gamma)} \otimes l_- \tag{3.19}$$

Notice that the inner products $((\cdot,\cdot))_{\mathscr{K}_+^{(\gamma)}}$ and $((\cdot,\cdot))_{\mathscr{K}_-^{(\gamma)}}$ in $\mathscr{K}_+^{(\gamma)}$ and $\mathscr{K}_-^{(\gamma)}$ are given by

$$\begin{aligned} ((\zeta, \xi))_{\mathscr{K}_+^{(\gamma)}} &= \sum_{i \in \mathbb{Z}^\nu} e^{\sigma |i|} (\zeta_i, \xi_i)_0^{(\gamma)} \\ ((\zeta, \xi))_{\mathscr{K}_-^{(\gamma)}} &= \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} (\zeta_i, \xi_i)_0^{(\gamma)} \end{aligned} \tag{3.20}$$

for $\zeta = (\zeta_i)_{i \in \mathbb{Z}^\nu}$, $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu}$. Thus for $\gamma \in [0, 1]$ the inclusions

$$\mathscr{H}_+^{(\gamma)} \subset \mathscr{K}_+^{(\gamma)} \subset \mathscr{H}_0^{(\gamma)} \subset \mathscr{K}_-^{(\gamma)} \subset \mathscr{H}_-^{(\gamma)}$$

hold.

In the rest of this section, we suppress $\gamma \in [0, 1]$ in the notation. Recall the definition $\beta(\xi)$ in (2.27). We prove (3.1) and (3.2) for our case.

Lemma 3.2. (a) For any $p \in \mathbb{N}$

$$\int_{\mathscr{H}_-} \|\xi\|_-^p \, d\mu(\xi) < \infty$$

(b) For any $p \in \mathbb{N} \cup \{0\}$

$$\int_{\mathscr{H}_-} \|\beta(\xi)\|_-^2 \, \|\xi\|_-^p \, d\mu(\xi) < \infty$$

Proof. We note that for any $m \in \mathbb{N}$ and $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu} \in \mathscr{H}_-$

$$\|\xi\|_-^{2m} = \sum_{i_1 \in \mathbb{Z}^\nu} \cdots \sum_{i_m \in \mathbb{Z}^\nu} \exp\left(-\sigma \sum_{k=1}^m |i_k|\right)\left(\prod_{k=1}^m |\xi_{i_k}|_-^2\right)$$

Using Fatou's lemma, Hölder's inequality and the regularity of Gibbs measures (Definition 2.2), we see that

$$\int_{\mathscr{H}_-} \|\xi\|_-^{2m} \, d\mu(\xi) \leqslant (k(m))^m$$

where the constant $k(m)$ in the above is given by

$$k(m) = \left(\sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|}\right) e^{\delta/m} \left(\int_E (\omega^2)^m e^{-A^* \omega^2} \lambda(d\omega)\right)^{1/m} < \infty$$

Here we have used the fact that $(|\omega|_-^{(\gamma)})^2 \leqslant \omega^2$ for any $\gamma \in [0, 1]$. This proves the part (a).

(b) By the Schwarz inequality and the part (a) of the lemma, it suffices to show that

$$\int_{\mathscr{H}_-} \|\beta\|_-^4 \, d\mu < \infty \tag{3.21}$$

Notice that for $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega_{\log}$

$$\|\beta(\xi)\|_-^2 = \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} \, |\beta_i(\xi)|_-^2$$

For given $\gamma \in [0, 1]$ it follows from (2.27) and (2.14) that for any $i \in \mathbb{N}$ and $\xi \in \Omega_{\log}$

$$|\beta_i(\xi)|_- \leqslant |\xi_i|_{L^2} + |\partial \tilde{P}(\xi_i)|_{L^2} + \sum_{j \in \mathbb{Z}^\nu} |\partial^i U(\xi_i, \xi_j; |i-j|)|_{L^2} \tag{3.22}$$

where $|\cdot|_{L^2}$ is the norm in $L^2([0, 1]; \mathbb{R}^d, d\tau)$. It follows from Assumption 2.1 (b)–(c) and the definition of $\Omega_{\log}$ in (2.20) that $\|\beta(\xi)\|_- < \infty$ for any $\xi \in \Omega_{\log}$. Since $\mu(\Omega_{\log}) = 1$, $\|\beta\|_-$ is defined μ-a.e. By the Schwarz inequality,

$$\int_{\mathscr{H}} \|\beta\|_-^4 \, d\mu \leqslant \left(\sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|}\right)\left(\sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} \int_{\mathscr{H}_-} |\beta_i|_-^4 \, d\mu\right) \tag{3.23}$$

By the regularity of μ,

$$\int |\xi_i|_{L^2}^4 \, d\mu(\xi) \leqslant M_1 \tag{3.24}$$

uniformly in $i \in \mathbb{Z}^\nu$. Let $d\nu_0$ be the Gaussian measure on E for which its characteristic function is given by

$$\int_E \exp(i(h, \omega)_{L^2}) \, d\nu_0(\omega) = \exp\{-\tfrac{1}{2}(h, (-\Delta_p + A^*)^{-1} h)_{L^2}\} \tag{3.25}$$

where $A^* > 0$ is the constant appeared in Definition 2.2. By using Assumption 2.1 (b), the regularity of μ (Definition 2.2) and the Fernique theorem,[28] we obtain that for sufficiently small $\delta > 0$

$$\begin{aligned}\int_E |\partial \tilde{P}(\xi_i)|_{L^2}^4 \, d\mu(\xi) &\leqslant c_1 \int_E e^{4\delta |\omega|_\alpha^2} e^{-A^*\omega^2/2} \lambda(d\omega)\\ &= c_1 \int_E e^{4\delta |\omega|_\alpha^2} \, d\nu_0(\omega)\\ &\leqslant M_2 \end{aligned} \tag{3.26}$$

uniformly in $i \in \mathbb{Z}^\nu$. Finally, we consider the last term in (3.22). We use Assumption 2.1 (d), the Hölder inequality, and the regularity of μ to obtain that

$$\begin{aligned}&\int_E \left(\sum_{j \in \mathbb{Z}^\nu} |\partial^i U(\xi_i, \xi_j)|_{L^2} \right)^4 d\mu(\xi)\\ &\leqslant \sum_{j_1 \in \mathbb{Z}^\nu} \cdots \sum_{j_4 \in \mathbb{Z}^\nu} \int_{\mathscr{H}_{..}} \left\{ \prod_{k=1}^4 \Psi(|i - j_k|)(|\xi_i|_{L^2} + |\xi_{j_k}|_{L^2}) \right\} d\mu(\xi)\\ &\leqslant M e^\delta \int_E |\omega|^4 \, e^{-A^*\omega^2} \lambda(d\omega)\\ &\leqslant M_3 \end{aligned} \tag{3.27}$$

uniformly in $i \in \mathbb{Z}^\nu$. The part (b) of the lemma follows from (3.22)–(3.27). ∎

We decompose the logarithmic derivative β of the Gibbs measure μ given in (2.27) as follows: For $\xi = (\xi_i) \in \Omega_{\log}$,

$$\beta(\xi) = \beta_1(\xi) + \beta_2(\xi) \tag{3.28}$$

where

$$\begin{aligned}\beta_1(\xi) &= (\beta_{1,i}(\xi))_{i \in \mathbb{Z}^\nu}\\ \beta_{1,i}(\xi) &= -A^\gamma \xi_i, \qquad i \in \mathbb{Z}^\nu \end{aligned} \tag{3.29}$$

and

$$\beta_2(\xi) = (\beta_{2,i}(\xi))_{i \in \mathbb{Z}^\nu}$$

$$\begin{aligned}\beta_{2,i}(\xi) &= -A^{-(1-\gamma)}\partial^i \tilde{P}(\xi_i) - A^{-(1-\gamma)} \sum_{j \neq i} \partial^i U(\xi_i, \xi_j; |i-j|) \\ &\equiv \beta_{2,i}^{(1)}(\xi_i) + \beta_{2,i}^{(2)}(\xi)\end{aligned} \tag{3.30}$$

In order to apply Theorem 3.1 to our case, we need to introduce a sequence $\{b_n : n \in \mathbb{N}\}$ which satisfies the conditions in Theorem 3.1. We employ the approximation method used in ref. 37.

Let $g: \mathbb{R} \to \mathbb{R}$ be a C^∞-function satisfying the following properties[31, 37]:

$$\begin{aligned}&\text{(a)} \quad g \text{ is an odd function: } g(t) = -g(-t) \\ &\text{(b)} \quad g(t) = t \text{ for } t \in (-1, 1) \\ &\text{(c)} \quad g \text{ is monotone increasing with } 0 \leqslant g'(t) \leqslant 1 \\ &\text{(d)} \quad g(t) \to 2 \text{ as } t \to \infty\end{aligned} \tag{3.31}$$

For each $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we put

$$g_n(t) := ng(t/n) \tag{3.32}$$

Next, we note that for $\gamma > 0$ the Hilbert space $\mathsf{H}_- = \mathsf{H}_-^{(\gamma)}$ introduced in (2.14) consists of generalized functions and so $P(\omega) = \int_0^1 P(\omega(\tau))\, d\tau$, $\omega \in \mathsf{H}_-$, is not defined in general. As in ref. 37, we introduce the mean operators (the Féjer operators) M_n, $n \in \mathbb{N}$, as follows: Let

$$\{e_l : e_l(\tau) = \exp(i2\pi l\tau),\ l \in \mathbb{Z}\}$$

be the orthonormal basis for $L^2([0, 1])$ consisting of eigenvectors of $A = (-\Delta_p + 1)$. For $\omega = (\omega^1, \omega^2, \dots, \omega^d) \in \mathsf{H}_0 \subset L^2([0, 1]; \mathbb{R}^d, d\tau)$, define partial sum operators S_k, $k \in \mathbb{N}$, by

$$S_k\omega = \left(\sum_{l_1=-k}^{k} (e_{l_1}, \omega^1)\, e_{l_1}, \dots, \sum_{l_d=-k}^{k} (e_{l_d}, \omega^d)\, e_{l_d} \right) \tag{3.33}$$

The mean operators (Féjer operators) M_n, $n \in \mathbb{N}$, on H_0 are defined by

$$M_n\omega = \frac{1}{n+1} \sum_{k=0}^{n} S_k(\omega), \qquad \omega \in \mathsf{H}_0 \tag{3.34}$$

Using these operators, let us define operators G_n: $\mathsf{H}_- \to \mathsf{H}_0$, $n \in \mathbb{N}$, by

$$G_n \omega = A M_n A^{-1} \omega \tag{3.35}$$

We collect useful properties of M_n, $n \in \mathbb{N}$, which we shall use in the sequel.

Lemma 3.3 (ref. 37, Lemma 4.2). Let M_n, $n \in \mathbb{N}$, be the mean operators defined as in (3.27). Then the following properties hold:

(a) (Féjer's theorem) For $\omega \in C([0, 1]; \mathbb{R}^d)$, $|M_n \omega - \omega|_u \to 0$ as $n \to \infty$.

(b) $\|M_n\|_{\mathscr{L}(L^2, L^2)} \leqslant 1$, $n \in \mathbb{N}$.

(c) $\|A M_n\|_{\mathscr{L}(L^2, L^2)} \leqslant \alpha_n$, where $\alpha_n = 1 + (2\pi n)^2$.

(d) $|M_n \omega|_u \leqslant \sqrt{d}\, |\omega|_u$ for any $\omega \in C([0, 1]; \mathbb{R}^d)$.

(e) $G_n \omega = M_n \omega$, $\omega \in L^2$, $n \in \mathbb{N}$.

Here we have used the abbreviated notation $L^2 = L^2([0, 1]; \mathbb{R}^d, d\tau)$.

One can obtain the above properties from ref. 38, Chapter 8. For the proof, we refer to ref. 37.

We are ready to introduce a sequence $\{b_n; n \in \mathbb{N}\}$ which approximates the logarithmic derivative β of μ. Let $\{a_n : a_n > 0, n \in \mathbb{N}\}$ be a sequence of positive numbers which satisfies the condition that for any $\alpha \in \mathbb{R}_+$

$$a_n \exp(e^{\alpha n^2}) \to 0 \qquad \text{as} \quad n \to \infty \tag{3.36}$$

Recall the decomposition of β in (3.28)–(3.30). For given $\varepsilon \in (0, 1/4)$ and a sequence $\{a_n : a_n > 0, n \in \mathbb{N}\}$ satisfying (3.36), we define a sequence of mappings $\{b_n : n \in \mathbb{N}\}$, b_n: $\mathscr{H}_- \to \mathscr{H}_-$ by

$$b_n = b_{1,n} + b_{2,n} \tag{3.37}$$

where for any $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu}$ (and $\gamma \in [0, 1]$)

$$\begin{aligned} b_{1,n}(\xi) &= (b_{1,n,i}(\xi))_{i \in \mathbb{Z}^\nu} \\ b_{1,n,i}(\xi) &= -(A^\gamma \exp(-a_n A^\varepsilon))\, \xi_i \end{aligned} \tag{3.38}$$

and

$$b_{2,n}(\xi) = (b_{2,n,i}(\xi))_{i \in \mathbb{Z}^\nu} \tag{3.39}$$

where for $d=1$

$$\begin{aligned} b_{2,n,i}(\xi) &= -A^{-(1-\gamma)} G_n \tilde{P}'(g_n(G_n\xi_i)) \\ &\quad - A^{-(1-\gamma)} \left(\sum_{j\neq i} G_n \partial^i U(G_n\xi_i, G_n\xi_j; |i-j|) \right) \\ &\equiv b_{2,n,i}^{(1)}(\xi_i) + b_{2,n,i}^{(2)}(\xi) \end{aligned} \tag{3.40}$$

and for $d\geq 2$

$$\begin{aligned} b_{2,n,i}(\xi) &= -A^{-(1-\gamma)} \{ G_n(\tilde{Q}'(g_n(|G_n\xi_i|))(G_n\xi_i/|G_n\xi_i|)) + b\} \\ &\quad - A^{-(1-\gamma)} \left\{ \sum_{j\neq i} G_n \partial^i U(G_n\xi_i, G_n\xi_j; |i-j|) \right\} \\ &\equiv b_{2,n,i}^{(1)}(\xi_i) + b_{2,n,i}^{(2)}(\xi) \end{aligned} \tag{3.41}$$

In the above, we have used the notation that for $x\in\mathbb{R}^\nu$

$$\begin{aligned} P(x) &= \tilde{P}(x) + \tfrac{1}{2}|x|^2, & d&=1 \\ P(x) &= \tilde{Q}(|x|) + \tfrac{1}{2}|x|^2 + b\cdot x, & d&\geq 2 \end{aligned} \tag{3.42}$$

and $\tilde{P}'$ and $\tilde{Q}'$ are derivatives of $\tilde{P}$ and $\tilde{Q}$ respectively. See (2.26) and Assumption 2.6 (c).

In the following, we prove that the sequence of mappings $\{b_n, n\in\mathbb{N}\}$ introduced in the above satisfies the conditions in Theorem 3.1. In the rest of this section, we suppose that the assumptions in Theorem 2.7 (a) hold and we suppress γ in the notations.

Lemma 3.4. Let $\{b_n : n\in\mathbb{N}\}$ be a sequence of mappings $b_n : \mathscr{H}_- \to \mathscr{H}_-$ defined through (3.37)–(3.41). Then the following properties hold:

(a) For each $n\in\mathbb{N}$, $b_n \in C^2_{\text{pol}}(\mathscr{H}_-, \mathscr{H}_-)$.

(b) For each $n\in\mathbb{N}$, there exists a constant $c(n)$ such that

$$((b_n(\xi), \xi))_- \leq c(n)(1+\|\xi\|_-^2), \qquad \xi\in\mathscr{H}_-$$

Proof. (a) Notice that for $\xi=(\xi_i)_{i\in\mathbb{Z}^\nu}\in\mathscr{H}_-$

$$\|b_n(\xi)\|_-^2 = \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}(|b_{1,n,i}(\xi)|_- + |b_{2,n,i}^{(1)}(\xi)|_- + |b_{2,n,i}^{(2)}(\xi)|_-)^2$$

By the factor $\exp(-a_n A^\varepsilon)$, we have that for any $i \in \mathbb{Z}^\nu$

$$|b_{1,n,i}(\xi)|_- \leqslant c(n)\, |\xi_i|_-$$

for some $c(n) > 0$. By the definition of g_n, $n \in \mathbb{N}$, in (3.32), it is easy to show that for any $i \in \mathbb{Z}^\nu$

$$|b^{(1)}_{2,n,i}(\xi)|_- \leqslant c(n)$$

Let Ψ be the function introduced in Assumption 2.1 (b). It follows from Assumption 2.6 (d) that for any $i, j \in \mathbb{Z}^\nu$

$$\begin{aligned} e^{-\sigma |i|} \Psi(|i-j|)\, e^{\sigma |j|} &\leqslant K e^{-\sigma |i-j|} \\ e^{-\sigma |i|/2} \Psi(|i-j|)\, e^{\sigma |j|/2} &\leqslant K e^{-\sigma |i-j|} \end{aligned} \tag{3.43}$$

The above inequalities will be used frequently in the sequel. Using (3.40) and (3.41), Assumption 2.1 (d), and the first inequalitiy in (3.43), we obtain that

$$\begin{aligned} &\sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|}\, |b^{(2)}_{2,n,i}(\xi)|^2_- \\ &\quad \leqslant c \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} \left(\sum_{j \in \mathbb{Z}^\nu} \Psi(|i-j|)(|G_n \xi_i|_{L^2} + |G_n \xi_j|_{L^2}) \right)^2 \\ &\quad \leqslant c' \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} \left(\sum_{j \in \mathbb{Z}^\nu} \Psi(|i-j|)(|G_n \xi_i|^2_{L^2} + |G_n \xi_j|^2_{L^2}) \right) \\ &\quad \leqslant K_1 \alpha_n^{2\gamma} \|\xi\|^2_- + \alpha_n^{2\gamma} \sum_{i, j \in \mathbb{Z}^\nu} (e^{-\sigma |i|} \Psi(|i-j|)\, e^{\sigma |j|})\, e^{-\sigma |j|}\, |G_n \xi_j|^2_{L^2} \\ &\quad \leqslant \alpha(n)\, \|\xi\|^2_- \end{aligned}$$

Here we have used the fact that $|G_n \xi|_{L^2} \leqslant \alpha_n^\gamma\, |\xi|_-$ by Lemma 3.3 (c). Combining the above results, we conclude that

$$\|b_n(\xi)\|^2_- \leqslant c(n)(1 + \|\xi\|_-)^2$$

for some constant $c(n) > 0$.

Let b'_n be derivative of b_n, $n \in \mathbb{N}$. Since $\|b'_n(\xi)\|_{\mathscr{L}(\mathscr{H}_-, \mathscr{H}_-)} \leqslant \|b'_n(\xi)\|_{\mathscr{L}(\mathscr{H}_-, \mathscr{K}_-)}$, it follows from Lemma 3.5 (a) given below that

$$\|b'_n(\xi)\|_{\mathscr{L}(\mathscr{H}_-, \mathscr{H}_-)} \leqslant c(n)$$

Using Assumption 2.6 (d) and the definition of b_n, $n \in \mathbb{N}$, one can also show that the operator norm of $b''_n(\xi)$ is bounded uniformly in $\xi \in \mathscr{H}_-$. This proves the part (a) of the lemma.

(b) Since the $|\cdot|_-$-norm of $b_{2,n,i}(\xi)$ is bounded uniformly in $\xi \in \mathscr{H}_-$, it is easy to show that

$$((b_{1,n}(\xi) + b^{(1)}_{2,n}(\xi), \xi))_- \leqslant c(n)(1 + \|\xi\|^2_-)$$

On the other hand, it follows from Assumption 2.1 (d) and the second inequality in (3.43) that

$$\begin{aligned}
&((b^{(2)}_{2,n}(\xi), \xi))_- \\
&\quad = \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} \Bigg(\sum_{j \in \mathbb{Z}^\nu:\, j \neq i} A^{-(1-\gamma)} G_n \partial^i U(G_n \xi_i, G_n \xi_j), \xi_i \Bigg)_- \\
&\quad \leqslant \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|} \Bigg\{ \sum_{j \in \mathbb{Z}^\nu} \Psi(|i-j|)(|G_n \xi_i|_{L^2} + |G_n \xi_j|_{L^2}) \Bigg\} |\xi_i|_- \\
&\quad \leqslant K \alpha_n^\gamma \Bigg\{ \|\xi\|^2_- + \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|/2} |\xi_i|_- \Bigg(\sum_{j \in \mathbb{Z}^\nu} e^{-\sigma |i-j|} e^{-\sigma |j|/2} |\xi_j|_- \Bigg) \Bigg\} \\
&\quad \leqslant c(n)\, \|\xi\|^2_-
\end{aligned}$$

The above inequalities imply the part (b) of the lemma. This completes the proof of the lemma. ∎

We next compute the derivative of b_n which will be used later. For any $\phi \in \mathscr{H}_-$, $\psi \in \mathscr{H}_+$,

$$\langle\langle b'_n(\xi)\, \phi, \psi \rangle\rangle = \lim_{t \to \infty} \frac{1}{t} \{ \langle\langle b_n(\xi + t\phi), \psi \rangle\rangle - \langle\langle b_n(\xi), \psi \rangle\rangle \}$$

Consider the case for $d = 1$. By a direct computation, it follows from (3.37)–(3.40) that for $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu} \in \mathscr{H}_-$

$$\begin{aligned}
b'_n(\xi) &= b'_{1,n}(\xi) + b'_{2,n}(\xi) \\
b'_{2,n}(\xi) &= b^{(1)'}_{2,n}(\xi) + b^{(2)'}_{2,n}(\xi)
\end{aligned} \tag{3.44}$$

where for $\phi = (\phi_i)_{i \in \mathbb{Z}^\nu} \in \mathscr{H}_-$

$$(b'_{1,n}(\xi)\,\phi)_i = -(A^\gamma \exp(-a_n A^\varepsilon))\,\phi_i$$
$$(b^{(1)'}_{2,n}(\xi)\,\phi)_i = -A^{-(1-\gamma)} G_n(\tilde{P}''(g_n(G_n\xi_i))\,g'_n(G_n(\xi_i))\,G_n\phi_i)$$
$$(b^{(2)'}_{2,n}(\xi)\,\phi)_i = \sum_{j \neq i} A^{-(1-\gamma)} G_n\{\partial^i\partial^i U(G_n\xi_i, G_n\xi_j)\,G_n\phi_i + \partial^i\partial^j U(G_n\xi_i, G_n\xi_j)\,G_n\phi_j\} \tag{3.45}$$

Next, consider the case for $d \geqslant 2$. For any $x = (x^1, x^2,..., x^d) \in \mathbb{R}^d$, denote by $\vec{x}\vec{x}$ the $d \times d$ matrix whose k-l th elements, $k, l = 1, 2,..., d$, are given by $x^k x^l$. A direct computation yields that

$$\text{Hess. } \tilde{P}(x) = \tilde{Q}''(|x|)\,\vec{x}\vec{x}/|x|^2 + \tilde{Q}'\left(\mathbf{1} - \frac{\vec{x}\vec{x}}{|x|^2}\right)\Big/|x| \tag{3.46}$$

See Assumption 2.6 (c). For each $n \in \mathbb{N}$, put

$$\tilde{R}_n(x) := -\tilde{Q}''(g_n(|x|))\,g'_n(|x|)\,\vec{x}\vec{x}/|x|^2 - \tilde{Q}'(g_n(|x|))\left(\mathbf{1} - \frac{\vec{x}\vec{x}}{|x|^2}\right)\Big/|x| \tag{3.47}$$

Recall the definition of $b_{2,n}(\xi)$ in (3.37), (3.39), and (3.41) for $d \geqslant 2$. In the case for $d \geqslant 2$, a computation gives that for $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu}$, $\phi = (\phi_i)_{i \in \mathbb{Z}^\nu} \in \mathscr{H}_-$,

$$\begin{aligned} b'_n(\xi) &= b'_{1,n}(\xi) + b'_{2,n}(\xi) \\ b'_{2,n}(\xi) &= b^{(1)'}_{2,n}(\xi) + b^{(2)'}_{2,n}(\xi) \\ (b'_{1,n}(\xi)\,\phi)_i &= -(A^\gamma \exp(-a_n A^\varepsilon))\,\phi_i \\ (b^{(1)'}_{2,n}(\xi)\,\phi)_i &= -A^{-(1-\gamma)} G_n(\tilde{R}_n(G_n\xi_i))\,G_n\phi_i \\ (b^{(2)'}_{2,n}(\xi)\,\phi)_i &= -\sum_{j \neq i} A^{-(1-\gamma)} G_n\{\partial^i\partial^i U(G_n\xi_i, G_n\xi_j)\,G_n\phi_i + \partial^i\partial^j U(G_n\xi_i, G_n\xi_j)\,G_n\phi_j\} \end{aligned} \tag{3.48}$$

From (3.45) and (3.48) we have the following result:

Lemma 3.5. (a) For any $n \in \mathbb{N}$, there exists a constant $M(n) \geqslant 0$ such that the bound

$$\|b'_n(\xi)\|_{\mathscr{L}(\mathscr{H}_-, \mathscr{H}_-)} \leqslant M(n)$$

holds uniformly in $\xi \in \mathscr{H}_-$.

(b) There exists a constant $\alpha > 0$ such that the bounds

$$((\phi, b_n'(\xi)\,\phi))_- \leqslant e^{\alpha n^2}\,\|\phi\|_-^2$$

hold uniformly in $\xi \in \mathscr{H}_-$.

Proof. (a) From the definition of $\mathscr{H}_-$ in (3.20), it follows that for any $\xi = (\xi_i)$, $\phi = (\phi_i) \in \mathscr{H}_-$

$$\|b_n'(\xi)\,\phi\|_{\mathscr{H}}^2 = \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma |i|}\, |A^{(1-\gamma)/2}(b_n'(\xi)\,\phi)_i|_{L^2}^2 \tag{3.49}$$

By the factor $\exp(-a_n A^\varepsilon)$ in $b_{1,n}'$, it follows from (3.45) that for each $n \in \mathbb{N}$

$$\begin{aligned} |A^{(1-\gamma)/2}(b_{1,n}'(\xi)\,\phi)_i|_{L^2} &\leqslant \|A^{(1-\gamma)/2} e^{-a_n A^\varepsilon} A^\gamma\|_{\mathscr{L}(L^2, L^2)}\, |\phi_i|_- \\ &\leqslant \|A^2 e^{-a_n A^\varepsilon}\|_{\mathscr{L}(L^2, L^2)}\, |\phi_i|_- \end{aligned}$$

We notice that $A \geqslant 1$ is an unbounded operator and the function $x \to x^2 e^{-a_n x^\varepsilon}$, $x \geqslant 1$, has its maximum value

$$M_1(n) := \left(\frac{2}{\varepsilon a_n}\right)^{2/\varepsilon} e^{-2/\varepsilon}$$

Thus, from (3.49) and above inequality, we see that

$$\|b_{1,n}'(\xi)\,\phi\|_{\mathscr{H}_-} \leqslant M_1(n)\,\|\phi\|_- \tag{3.50}$$

Next, we consider $b_{2,n}$. From the properties of g_n listed in (3.31) and Lemma 3.3, one can easily check that for any $n \in \mathbb{N}$, the bounds

$$\begin{gathered} \|A^p G_n\|_{\mathscr{L}(L^2, L^2)} \leqslant \alpha(n)^p \\ |g_n(|G_n \omega|)|_u \leqslant 2n \\ |g_n'(|G_n \omega|)|_u \leqslant 1 \\ |\tilde{P}'(g_n(|G_n\omega|))|_u + |\tilde{P}''(g_n(|G_n\omega|))|_u \leqslant M(\delta)\exp(4\delta n^2) \end{gathered} \tag{3.51}$$

hold uniformly in $\omega \in \mathrm{H}_-$. For the last bound in the above, we have used Assumption 2.1 (b) and Assumption 2.6 (a). We first consider the case for $d = 1$. Using the expression in (3.45) and the bounds in (3.51), one obtains that

$$|A^{(1-\gamma)/2}(b_{2,n}^{(1)\prime}(\xi)\,\phi)_i|_{L^2} \leqslant e^{ln^2}\,|\phi_i|_-$$

for some constant $l \geqslant 0$, and so by (3.49)

$$\|b_{2,n}^{(1)'}(\xi)\,\phi\|_{\mathscr{K}_-} \leqslant e^{ln^2}\,\|\phi\|_- \tag{3.52}$$

Due to Assumption 2.6 (d), the first inequality in (3.43), and Lemma 3.3 (c), a direct estimate yields that

$$\begin{aligned}\|b_{2,n}^{(2)'}(\xi)\,\phi\|^2_{\mathscr{K}_-} &\leqslant \alpha(n)^2 \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|} \left(\sum_{j \neq i} \Psi(|i-j|)(|\phi_i|_- + |\phi_j|_-)\right)^2 \\ &\leqslant K'\alpha(n)^2\,\|\phi\|^2_- \end{aligned}\tag{3.53}$$

for some constant K'. Thus, the part (a) of the lemma for $d=1$ follows from (3.50), (3.52) and (3.53).

Consider the case for $d \geqslant 2$. Let $\tilde{R}_n(x)$ be defined as in (3.47). Note that

$$\tilde{R}_n(x) = \text{Hess.}\,\tilde{P}(x) \qquad \text{if} \quad |x| \leqslant n$$

Since $\tilde{P}$ is a C^3-function on $\mathbb{R}^d$ and since $|g_n(|x|)| \leqslant 2n$ for any $x \in \mathbb{R}^d$, by Assumption 2.6 (d) there exists a constant $c \geqslant 0$ such that

$$\|\tilde{R}_n(x)\|_{\mathscr{L}(\mathbb{R}^d,\,\mathbb{R}^d)} \leqslant e^{cn^2} \tag{3.54}$$

Using (3.54), the definition $b_{2,n}^{(1)'}$ in (3.48), (3.49), and the bounds in (3.51), it is easy to show that

$$\|b_{2,n}^{(1)'}(\xi)\,\phi\|_{\mathscr{K}_-} \leqslant e^{ln^2}\,\|\phi\|_- \tag{3.55}$$

for some constant $l \geqslant 0$ uniformly in $\xi \in \mathscr{H}_-$. Thus the part (a) of the lemma for $d \geqslant 2$ follows from (3.50), (3.55), and (3.53).

(b) Notice that for any $\xi = (\xi_i)$, $\phi = (\phi_i) \in \mathscr{H}_-$ and $n \in \mathbb{N}$

$$((\phi, b'_n(\xi)\,\phi))_- = \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|}(\phi_i, (b'_n(\xi)\,\phi)_i)_-$$

From the definition of $b'_{1,n}$ in (3.44), one has that

$$(\phi_i, b'_{1,n}\phi_i)_- \leqslant 0$$

We notice that

$$(\phi_i, b_{2,n}^{(1)'}(\xi)\,\phi_i)_- = (\phi_i, A^{-2\gamma} b_{2,n}^{(1)'}(\xi)\,\phi_i)_{L^2}$$

By using the bounds in (3.51) and the method similar to that employed to obtain (3.52) and (3.55), one can show that for each $n \in \mathbb{N}$ there exists a constant $l \geqslant 0$ such that

$$(\phi_i, b_{2,n}^{(1)\prime}(\xi)\,\phi_i)_- \leqslant e^{ln^2}\,|\phi_i|_-^2$$

Thus we conclude that

$$((\phi, b_{1,n}\phi))_- + ((\phi, b_{2,n}^{(1)}\phi))_- \leqslant e^{ln^2}\,\|\phi\|_-$$

By using the method similar to that used to obtain (3.53), it is easy to show that for each $n \in \mathbb{N}$ there exists a constant K' such that

$$((\phi, b_{2,n}^{(2)\prime}(\xi)\,\phi))_- \leqslant K'\alpha(n)^2\,\|\phi\|_-^2$$

The part (b) of the lemma follows from the above bounds. ∎

Lemma 3.6. There exists a constant $c_1 \geqslant 0$ and $N_0 \in \mathbb{N}$ such that for any $n \geqslant N_0$ and $h \in \mathscr{K}_-$ the bound

$$((h, b_n'(\xi)\,h))_{\mathscr{K}_-} \leqslant c_2\,\|h\|_{\mathscr{K}_-}^2$$

holds uniformly in $\xi \in \mathscr{H}_-$ and $n \geqslant N_0$.

Proof. It follows from the definition of $\mathscr{K}_-$ in (3.20) that for any $n \in \mathbb{N}$, $\xi = (\xi_i) \in \mathscr{H}_-$ and $h = (h_i) \in \mathscr{K}_-$

$$((h, b_n'(\xi)\,h))_{\mathscr{K}_-} = \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|}(h_i, (b_n'(\xi)\,h)_i)_0 \tag{3.56}$$

Consider the case for $d = 1$. From (3.45) and (2.14) it follows that

$$\begin{aligned}(h_i, (b_{1,n}'(\xi)\,h)_i)_0 &= -(h_i, A\exp(-a_n A^\varepsilon)\,h_i)_{L^2}\\ &\leqslant 0\end{aligned} \tag{3.57}$$

Recall that $0 \leqslant g_n'(x) \leqslant 1$ by (3.32), and $\tilde{P}(x) = P(x) - |x|^2/2$. Thus by Assumption 2.6 (b) we obtain that

$$\begin{aligned}(h_i, (b_{2,n}^{(1)\prime}(\xi)\,h)_i)_0 &= -(h_i, G_n\tilde{P}''(g_n(G_n\xi_i))\,g_n'(G_n\xi_i)\,G_nh_i)_{L^2}\\ &\leqslant (-M+1)\,|h_i|_{L^2}^2\\ &\leqslant (-M+1)\,|h_i|_0^2\end{aligned} \tag{3.58}$$

Using Assumption 2.6 (d), we also obtain that

$$((h, b^{(2)'}_{2,n}(\xi)\,h))_{\mathscr{H}_-} = \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|}(h_i, (b^{(2)'}_{2,n}(\xi)\,h)_i)_0$$

$$\leqslant \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|} \left\{ \sum_{j \neq i} \Psi(|i-j|)(|h_i|_0 + |h_j|_0) \right\} |h_i|_0 \tag{3.59}$$

We use the second inequality in (3.43) to (3.59) to conclude that

$$((h, b^{(2)'}_{2,n}(\xi)\,h))_{\mathscr{H}} \leqslant K\Bigg(\|h\|^2_{\mathscr{H}_-} + \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|/2}\,|h_i|_0 \times \left\{ \sum_{j \neq i} e^{-\sigma|i-j|} e^{-\sigma|j|/2}\,|h_j|_0 \right\}\Bigg)$$

$$\leqslant c'\,\|h\|^2_{\mathscr{H}_-} \tag{3.60}$$

for some constant $c' \geqslant 0$. The lemma for $d=1$ follows from (3.56)–(3.58) and (3.60).

We next consider for $d \geqslant 2$. For given $\varepsilon > 0$, put $M_\varepsilon := M - \varepsilon$. We write

$$P(x) = Q(|x|) + b \cdot x = b \cdot x + \tfrac{1}{2} M_\varepsilon\,|x|^2 + \hat{Q}(|x|)$$

where M is the constant appeared in Assumption 2.6 (b). Since $P(x) = \tilde{Q}(|x|) + \frac{1}{2}|x|^2 + b \cdot x$ (see (3.42)),

$$\tilde{Q}(|x|) = \tfrac{1}{2}(M_\varepsilon - 1)\,|x|^2 + \hat{Q}(|x|)$$

Since Hess. $P(x) \geqslant M\mathbf{1}$, we see that Hess. $\hat{Q}(|x|) \geqslant \varepsilon\mathbf{1}$. Thus we may assume that there exists $R > 0$ such that

$$\hat{Q}'(|x|) \geqslant 0 \qquad \text{and} \qquad \hat{Q}''(|x|) \geqslant 0 \text{ if } |x| \geqslant R \tag{3.61}$$

Let $\tilde{R}_n(x)$ be defined as in (3.47). Suppose that n is sufficently large so that $n \geqslant R$. Then, we note that for any $x, y \in \mathbb{R}^d$

$$(y, \tilde{R}_n(x)\,y) = \left(y, \left\{ (\hat{Q}''(g_n(|x|)) + (M_\varepsilon - 1))\,g_n'(|x|)\,\frac{\vec{x}\vec{x}}{|x|^2}\,y \right\} \right)$$

$$+ \left(y, \left\{ (\hat{Q}'(g_n(|x|)) + (M_\varepsilon - 1))\,g_n(|x|) \left(\mathbf{1} - \frac{\vec{x}\vec{x}}{|x|^2} \right) \Big/ |x| \right\} y \right)$$

where (x, y) is the inner product of x and y in $\mathbb{R}^d$. We note that $\vec{x}\vec{x}/|x|^2$ and $(\mathbf{1}-(\vec{x}\vec{x}/|x|^2))/|x|$ are positive definite for any $0 \neq x \in \mathbb{R}^d$. Thus, we conclude that

$$(y, \tilde{R}_n(x)\, y) \geqslant (M_\varepsilon - 1)\, |y|^2, \qquad \text{if} \quad |x| \leqslant R$$

and

$$\begin{aligned}(y, \tilde{R}_n(x)\, y) &\geqslant (M_\varepsilon - 1)\left(y, g_n'(|x|)\,\frac{\vec{x}\vec{x}}{|x|^2}\, y + g_n(|x|)\left(\left(\mathbf{1} - \frac{\vec{x}\vec{x}}{|x|^2}\right)\Big/|x|\right) y\right) \\ &\geqslant \min\{0, M_\varepsilon - 1\}\, |y|^2, \qquad \text{if} \quad |x| > R \qquad (3.62)\end{aligned}$$

In the above we have used the fact that $g_n(|x|) = |x|$, if $|x| \leqslant R$ ($\leqslant n$) (and hence $(y, \tilde{R}_n(x)\, y) = (y, \text{Hess. } \tilde{V}(x)\, y) \geqslant (M-1)\, |y|^2$), and $g_n(|x|) \geqslant R$ if $|x| > R$. Using the above bound, we obtain that for $\xi = (\xi_i) \in \mathscr{H}_-$ and $h = (h_i) \in \mathscr{H}_-$

$$\begin{aligned}(h_i, b_{2,n,i}^{(1)'}(\xi)\, h_i)_0 &= -(h_i, G_n \tilde{R}_n(G_n \xi_i)\, G_n h_i)_{L^2} \\ &\leqslant (|M| + 1)\, |h_i|_{L^2} \\ &\leqslant (|M| + 1)\, |h_i|_0\end{aligned}$$

Thus by (3.56), we conclude that

$$((h, b_{2,n}^{(1)'} h))_{\mathscr{H}_-} \leqslant (|M| + 1)\, \|h\|_{\mathscr{H}_-} \qquad (3.63)$$

The part (b) of the lemma follows from (3.57), (3.60), and (3.63). This completes the proof of the lemma. ∎

Finally, we show that the conditions (vi) and (vii) of Theorem 3.1 are satified:

Lemma 3.7. Under the assumptions in Theorem 2.7 (a) the following results hold:

(a) There exists a constant c such that for each $n \in \mathbb{N}$ the bound

$$\|\, \|\beta_1 - b_{1,n}\|_- \,\|_{L^2(\mu)} \leqslant c a_n$$

holds, where $\{a_n : n \in \mathbb{N}\}$ is the sequence introduced in the definition of $b_{1,n}$.

(b)

$$\|\, \|\beta_2 - b_{2,n}\|_{\mathscr{H}_-} \,\|_{L^2(\mu)} \to 0 \qquad \text{as} \quad n \to \infty$$

Proof. Due to the definitions of β_1 and $b_{1,n}$ in (3.29) and (3.38) respectively, we have that for $\xi = (\xi_i) \in \mathscr{H}_-$

$$\|\beta_1(\xi) - b_{1,n}(\xi)\|_-^2 = \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|}(\xi_i, (1 - \exp(-a_n A^\varepsilon))^2 \xi_i)_{L^2}$$

$$\leqslant a_n^2 \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|}(\xi_i, A^{2\varepsilon}\xi_i)_{L^2}$$

Here we have used the fact $1 - \exp(-x) \leqslant x$ for $x \geqslant 0$. Notice that for any $\varepsilon \in (0, 1/4)$, the operator $A^{2\varepsilon - 1}$ belongs to the trace class. Let $d\nu_0$ be the Gaussian measure given in (3.25). Then by the regularity of μ and Theorem 3.11 of ref. 46 we conclude that for given $\varepsilon \in (0, 1/4)$

$$\int_E (\xi_i, A^{2\varepsilon}\xi_i)_{L^2}\, d\mu(\xi) \leqslant \text{const.} \int_E (\omega, A^{2\varepsilon}\omega)\, d\nu_0(\omega)$$

$$< \infty$$

This proves the part (a) of the lemma.

(b) We first consider the case for $d = 1$. Let $\beta_2^{(1)}, \beta_2^{(2)}$ resp. $b_{2,n}^{(1)}, b_{2,n}^{(2)}$ be given as in (3.30) resp. (3.40). Then for $\xi = (\xi_i) \in \mathscr{H}_-$

$$\|\beta_2(\xi) - b_{2,n}(\xi)\|_{\mathscr{K}_-}^2 \leqslant 2 \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|}\{|\beta_{2,n}^{(1)}(\xi_i) - b_{2,n,i}^{(1)}(\xi_i)|_0^2$$

$$+ |\beta_{2,i}^{(2)}(\xi) - b_{2,n}^{(2)}(\xi)|_0^2\} \tag{3.64}$$

Let us consider the first term in the right hand side of (3.64). For any $\omega \in E$ it follows from (3.30) and (3.40) that

$$|\beta_{2,i}^{(1)}(\omega) - b_{2,n,i}^{(1)}(\omega)|_0^2 = \int_0^1 [A^{(1-\gamma)/2}(\beta_{2,i}^{(2)}(\omega(\tau) - b_{2,n,i}^{(2)}(\omega(\tau)))]^2\, d\tau$$

$$\leqslant \int_0^1 [\tilde{P}'(\omega(\tau)) - G_n \tilde{P}'(g_n(G_n(\omega)))]^2\, d\tau$$

By Lemma 3.3 (d)–(e) and Assumption 2.1 (b), we obtain that

$$|\beta_{2,i}^{(1)}(\omega) - b_{2,n,i}^{(1)}(\omega)|_0^2 \leqslant 4M(\delta) \exp(4\delta\sqrt{d}\, |\omega|_u^2) \tag{3.65}$$

Let $d\nu_0$ be the Gaussian measure on E defined in (3.25). Then the Fernique theorem[28] implies that for sufficently small $\delta > 0$

$$\int \exp(4\delta \sqrt{d}\, |\omega|_u^2)\, d\nu_0(\omega) < \infty$$

Using the monotone convergence theorem, the regularity of μ, the Lebesgue dominated convergence theorem and Lemma 3.3 (a), we conclude that

$$\| \|\beta_2^{(1)} - b_{2,n}^{(1)}\|_{\mathscr{K}_-} \|_{L^2(\mu)}^2 \leqslant \text{const.} \int_E [\tilde{P}'(\omega(\tau)) - G_n \tilde{P}'(g_n(M_n \omega))]^2\, d\nu_0(\omega)$$

$$\to 0 \qquad \text{as} \quad n \to \infty \tag{3.66}$$

Next, let us consider the second term in the right hand side of (3.64). By Assumption 2.1 (d),

$$|\beta_{2,i}^{(2)}(\xi) - b_{2,n,i}^{(2)}(\xi)|_0^2 \leqslant \text{const.} \sum_{j \neq i} \Psi(|i-j|)(|\xi_i|_u^2 + |\xi_j|_u^2)$$

which is μ-integrable by the regularity of μ, the Fernique theorem and Assumption 2.6 (d). By Lemma 3.3 (a), one can show that for any $\xi = (\xi_i) \in \Omega_{\log}$, $i, j \in \mathbb{Z}^\nu$, and $\tau \in [0, 1]$

$$\partial^i U(\xi_i(\tau), \xi_j(\tau)) - G_n \partial^i U(G_n \xi_i(\tau), G_n \xi_j(\tau)) \to 0 \qquad \text{as} \quad n \to \infty$$

Thus we use again the monotone convergence theorem and the Lebesgue dominated convergence theorem to conclude that

$$\| \|\beta_2^{(2)} - b_{2,n}^{(2)}\|_{\mathscr{K}_-} \|_{L^2(\mu)} \to 0 \qquad \text{as} \quad n \to \infty \tag{3.67}$$

Combining (3.66) and (3.67), we complete the proof of the lemma for $d = 1$.

For $d \geqslant 2$, we need only to show (3.66). By Assumption 2.1 (b) and Lemma 3.3 (d)–(e), it can be checked that the bound in (3.65) also holds in the case of $d \geqslant 2$. Thus by the method used to obtain (3.66), we conclude that (3.66) holds for $d \geqslant 2$. This completes the part (b) of the lemma. ∎

We are now in a position to prove Theorem 2.7.

Proof of Theorem 2.7. (a) We need to check that all conditions in Theorem 3.1 are satisfied. The conditions (3.1) and (3.2) are satisfied by Lemma 3.2. The conditions (i)–(v) in Theorem 3.1 are satisfied by Lemma 3.4, Lemma 3.5, and Lemma 3.6. Notice that we have chosen the sequence $\{a_n : n \in \mathbb{N}\}$ such that the property (3.36) holds, and so Lemma 3.5 (b) and Lemma 3.7 (a) imply that the condition (vi) is satisfied. Finally Lemma 3.7 (b) implies the condition (vii). Thus Theorem 2.7 (a) follows from Theorem 3.1.

(b) Recall the definition of $\mathscr{F}_{\mathrm{loc}} C_b^k$ in (2.24), i.e., $\mathscr{F}_{\mathrm{loc}} C_b^k = \mathscr{F}_{\mathscr{K}_0} C_b^k$. Since $\mathscr{K}_0$ is dense in $\mathscr{H}_+$, the part (b) of the theorem follows from Lemma 6 of ref. 7 together with the part (a) of the theorem. For the details, we refer to Lemma 6 of ref. 7 and its proof. This proves Theorem 2.7 completely. ∎

4. UNIFORM LOG-CONCAVITY AND LOG-SOBOLEV INEQUALITY

We shall discuss the uniform log-concavity (ULC) and log-Sobolev inequality (LS) for Gibbs measures, and then produce the proofs of Theorem 2.10 and Theorem 2.12. Recall the definition of $R_\mu^{(\gamma)}$ in (2.30). Throughout this section, $\gamma \in [0, 1]$ is given (fixed) and we again suppress γ in the notation. We also recall the definition of $\Omega_{\log}$ in (2.20) and the fact that $\mu(\Omega_{\log}) = 1$ for any $\mathscr{G}^{\Phi}(\Omega)$. It follows from (2.27) and (2.30) that for any $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega_{\log}$ and $\phi = (\phi_i)_{i \in \mathbb{Z}^\nu} \in \mathscr{H}_+$

$$\begin{aligned}(R_\mu(\xi)\,\phi)_i &= A^\gamma \phi_i + A^{-(1-\gamma)}(\mathrm{Hess.}\,\tilde{P}(\xi_i)\,\phi_i) \\ &\quad + \sum_{j \neq i} A^{-(1-\gamma)}\{(\partial^i \partial^i U(\xi_i, \xi_j))\,\phi_i + \partial^i \partial^j U(\xi_i, \xi_j))\,\phi_j\} \\ &\equiv (R_{\mu,1}\phi)_i + (R_{\mu,2}^{(1)}(\xi)\,\phi)_i + (R_{\mu,2}^{(2)}(\xi)\,\phi)_i \end{aligned} \tag{4.1}$$

We begin with the following result:

Lemma 4.1. Under Assumption 2.1 and Assumption 2.6, $R_\mu(\xi) \in \mathscr{L}(\mathscr{H}_+, \mathscr{H}_-)$ for any $\xi \in \Omega_{\log}$.

Proof. Let $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega_{\log}$. Then there is $N \in \mathbb{N}$ such that $|\xi_i|_u \leqslant N \log(|i| + 1)$, $i \in \mathbb{Z}^\nu$. By (4.1) we have that for $\phi = (\phi_i) \in \mathscr{H}_+$

$$|(R_{\mu,1}\phi)_i|_-^2 = |\phi_i|_{L^2}^2 \leqslant |\phi_i|_+^2 \tag{4.2}$$

By Assumption 2.6 (a) and (4.1), we also have that

$$|(R^{(1)}_{\mu,2}(\xi)\,\phi)_i|^2_- \leqslant M_1 e^{\delta\,|\xi_i|^2_u}\,|\phi_i|^2_{L^2}$$
$$\leqslant M_1(|i|+1)^{cN^2}\,|\phi_i|^2_+ \tag{4.3}$$

for some constants $M_1 \geqslant 0$ and $c \geqslant 0$. By using Assumption 2.6 (d) it is easy to show that

$$|(R^{(2)}_{\mu,2}(\xi)\,\phi)_i|^2_- \leqslant M_2 \sum_{j\neq i} \Psi(|i-j|)(|\phi_i|^2_+ + |\phi_j|^2_+) \tag{4.4}$$

By (4.2)–(4.4) and the first inequality in (3.43), it is clear that

$$\|R_\mu(\xi)\,\phi\|^2_- = \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}\,|(R_\mu(\xi)\,\phi)_i|^2_-$$
$$\leqslant M_3 \sum_{i\in\mathbb{Z}^\nu} e^{\sigma|i|}\,|\phi_i|^2_+$$
$$= M_3\,\|\phi\|^2_+$$

This completes the proof of the lemma. ∎

We now turn to prove Theorem 2.10.

Proof of Theorem 2.10. Recall the representation $(R_\mu(\xi))_i$ in (4.1). For $\xi=(\xi_i)\in\Omega_{\log}$ and $\phi=(\phi_i)\in\mathscr{H}_+$, we have that

$$\langle\!\langle R_{\mu,1}(\xi)\,\phi, \phi\rangle\!\rangle = \sum_{i\in\mathbb{Z}^\nu} \langle R_{\mu,1}\phi_i, \phi_i\rangle$$
$$= \sum_{i\in\mathbb{Z}^\nu} (A\phi_i, \phi_i)_{L^2} \tag{4.5}$$

and

$$\langle\!\langle R^{(1)}_{\mu,2}(\xi)\,\phi, \phi\rangle\!\rangle = \sum_{i\in\mathbb{Z}^\nu} \langle R^{(2)}_{\mu,2}(\xi_i)\,\phi_i, \phi_i\rangle$$
$$= \sum_{i\in\mathbb{Z}^\nu} ((\text{Hess.}\,\tilde{P})(\xi)\,\phi_i, \phi_i)_{L^2} \tag{4.6}$$

By Assumption 2.6 (d),

$$
\begin{aligned}
&|\langle\!\langle R^{(2)}_{\mu,2}(\xi)\,\phi, \phi\rangle\!\rangle| \\
&\quad = \sum_{i\in\mathbb{Z}^\nu} |\langle (R^{(2)}_{\mu,2}(\xi)\,\phi)_i, \phi_i\rangle| \\
&\quad \leqslant \sum_{i\in\mathbb{Z}^\nu} |\phi_i|_{L^2} \sum_{j\neq i} (|\partial^i\partial^i U(\xi_i, \xi_j; |i-j|)\,\phi_i|_{L^2} + |\partial^i\partial^j U(\xi_i, \xi_j; |i-j|)\,\phi_j|_{L^2}) \\
&\quad \leqslant d \sum_{i\in\mathbb{Z}^\nu} |\phi_i|_{L^2} \sum_{j\neq i} \Psi(|i-j|)(|\phi_i|_{L^2} + |\phi_j|_{L^2}) \\
&\quad \leqslant M' \sum_{i\in\mathbb{Z}^\nu} |\phi_i|^2_{L^2} \qquad (4.7)
\end{aligned}
$$

where M' is the constant appeared in Theorem 2.10. Since Hess. $P(x) \geqslant M\mathbf{1}$ (and so Hess. $\tilde{P}(x) \geqslant (M-1)\,\mathbf{1}$) and $A = -\Delta_p + \mathbf{1}$, it follows from (4.5)-(4.7) that for any $\xi \in \Omega_{\log}$ and $\phi \in \mathscr{H}_+$

$$
\begin{aligned}
\langle\!\langle R_\mu(\xi)\,\phi, \phi\rangle\!\rangle &\geqslant \sum_{i\in\mathbb{Z}^\nu} ((-\Delta_p + M - M')\,\phi_i, \phi_i)_{L^2} \\
&\geqslant \lambda\, \|\phi\|_0^2
\end{aligned}
$$

where $\lambda = \min\{M-M', (M-M')^\gamma\}$. Here we have used the fact that

$$(-\Delta_p + M - M')\,A^{-(1-\gamma)} \geqslant \min\{(M-M'), (M-M')^\gamma\}$$

This proves Theorem 2.10 completely. ∎

The rest of this section is devoted to prove the log-Sobolev inequality (Theorem 2.12). As in refs. 9 and 37, we can easily check that for any $\gamma \in [0, 1]$ and $u \in \mathscr{F}_{\mathrm{loc}} C_b^\infty(\mathscr{H}_-^{(\gamma)})$

$$\mathscr{E}_\mu^{(0)}(u, u) \leqslant \mathscr{E}_\mu^{(\gamma)}(u, u) \qquad (4.8)$$

Thus, it suffices to prove Theorem 2.12 for $\gamma = 0$. In the rest of this section, we consider only the case for $\gamma = 0$, and suppress $\gamma (=0)$ in the notation. Thus, $\mathscr{E}_\mu$, $\mathscr{H}_s$, H_s, $s \in \{+, 0, -\}$, stand for $\mathscr{E}_\mu^{(0)}$, $\mathscr{H}_s^{(0)}$, $\mathsf{H}_s^{(0)}$, $s \in \{+, 0, -\}$, etc.

We shall use an extended version of the method developed in the proof of Theorem 5.2 of ref. 37. Let μ be the unique Gibbs measure. In ref. 35, we have shown that for any pure boundary condition $\bar{\xi} \in \mathscr{S}$, the sequence $\{\mu_n\}$, $\mu_n = \gamma^\Phi_{\Lambda_n}(\cdot\,|\,\bar{\xi})$ (see (2.12)), has a limit point in $\mathscr{G}^\Phi(\Omega)$ for any sequence of finite subsets $\Lambda_n \subset \mathbb{Z}^\nu$, $\Lambda_n \uparrow \mathbb{Z}^\nu$. Since the set $\mathscr{G}^\Phi(\Omega)$ has only one element, we may assume that $\mu = \lim_{n \to \mathbb{Z}^\nu} \mu_n$, with a fixed pure boundary condition $\bar{\xi} \in \Omega_{\log}$.

By Theorem 2.7 (b), the Dirichlet operator H_μ is essentially self-adjoint on $\mathscr{F}_{\rm loc} C_b^\infty(\mathscr{H}_-)$. Thus it suffices to prove the log-Sobolev inequality for $f \in \mathscr{F}_{\rm loc} C_b^\infty(\mathscr{H}_-)$. Let us fix $f \in \mathscr{F}_{\rm loc} C_b^\infty(\mathscr{H}_-)$. Then there exist $\Delta \in \mathscr{C}$ such that f depends only on the variables in Δ. For any $\Lambda_n \supset \Delta$, we define

$$\int_{E^{\Lambda_n}} f(\omega_{\Lambda_n})\, d\mu^{(n)}(\omega_{\Lambda_n})$$
$$= Z_{\Lambda_n}(\bar{\xi})^{-1} \int \lambda(d\omega_{\Lambda_n}) \exp[-V(\omega_\Lambda) - W(\omega_{\Lambda_n}, \bar{\xi}_{\Lambda^c})]\, f(\omega_{\Lambda_n}) \tag{4.9}$$

See (2.11) and (2.12) for the notation. We consider $\mu^{(n)}$ as a measure on $(E^{\Lambda_n}, \mathscr{B}(E^{\Lambda_n}))$ defined by the formula (4.9).

For each $i \in \mathbb{Z}^\nu$ and $s \in \{+, 0, -\}$, let $\mathsf{H}_{s,i}$ be the identical copy of the Hilbert space H_s defined in (2.14)–(2.15) for $\gamma = 0$, and let

$$\begin{aligned} \mathsf{H}_s^{(n)} &= \bigoplus_{i \in \Lambda_n} \mathsf{H}_{s,i}, \qquad s \in \{+, 0, -\} \\ ((\omega, \xi))_{\mathsf{H}_s^{(n)}} &= \sum_{i \in \Lambda_n} e^{s\sigma |i|} (\omega_i, \xi_i)_{\mathsf{H}_s}, \qquad \omega, \xi \in \mathsf{H}_s^{(n)} \end{aligned} \tag{4.10}$$

By the definition, the embeddings

$$\mathsf{H}_+^{(n)} \subset \mathsf{H}_0^{(n)} \subset \mathsf{H}_-^{(n)} \tag{4.11}$$

are everywhere dense and belong to the Hilbert–Schmidt class. Then the logarithmic derivative $\beta^{(n)}$ of $\mu^{(n)}$ which is considered as a measure on $\mathscr{B}(\mathsf{H}_-^{(n)})$ can be easily calculated to be (cf. (2.27) for $\gamma = 0$)

$$\begin{aligned} \beta^{(n)}(\omega) &= \beta_1^{(n)}(\omega) + \beta_2^{(n,1)}(\omega) + \beta_2^{(n,2)}(\omega) \\ (\beta_1^{(n)}(\omega))_i &= -\omega_i \\ (\beta_2^{(n,1)}(\omega))_i &= -A^{-1} \partial \tilde{P}(\omega_i) \\ (\beta_2^{(n,2)}(\omega))_i &= -\sum_{j \in \Lambda_n :\, j \neq i} A^{-1} \partial^i U(\omega_i, \omega_j) - \sum_{j \in \Lambda_n^c} A^{-1} \partial^i U(\omega_i, \bar{\xi}_j) \end{aligned} \tag{4.12}$$

where $i \in \Lambda_n$ and $\omega = (\omega_i)_{i \in \Lambda_n} \in E^{\Lambda_n}$.

Let us describe the basic idea of the proof of Theorem 2.12. For each $n \in \mathbb{N}$, we shall approximate $\beta^{(n)}$ by a sequence of maps $\{b_m^{(n)} : m \in \mathbb{N}\}$, $b_m^{(n)} : \mathsf{H}_-^{(n)} \to \mathsf{H}_-^{(n)}$, $m \in \mathbb{N}$, which are logarithmic derivatives of measures $\mu_m^{(n)}$,

$m \in \mathbb{N}$, on E^{Λ_n}. We then show that for $n, m \in \mathbb{N}$, the measure $\mu_m^{(n)}$ satisfies the log-Sobolev inequality with Sovolev coefficient $c_{\mu_m^{(n)}} = \tilde{\lambda}^{-1}$ uniformly in $n, m \in \mathbb{N}$, and that for $n \in \mathbb{N}$, $\mu_m^{(n)}$ converges to $\mu^{(n)}$ weakly as $m \to \infty$. This implies that for $n \in \mathbb{N}$, $\mu^{(n)}$ satisfies the log-Sobolev inequality with $c_{\mu^{(n)}} = \tilde{\lambda}^{-1}$. Since $\mu^{(n)}$ converges to μ in the local convergence topology, we prove Theorem 2.12.

Let $\varepsilon > 0$ be an arbitrary (fixed) real number such that $M_\varepsilon - M' > 0$, where $M_\varepsilon := M - \varepsilon$, and M and M' are the constants appeared in Theorem 2.10. As in the proof of Lemma 3.6, we write the one-body potential P as

$$\begin{aligned} &P(x) = \tfrac{1}{2} M_\varepsilon |x|^2 + \hat{P}(x), && d = 1 \\ &P(x) = b \cdot x + \tfrac{1}{2} M_\varepsilon |x|^2 + \hat{Q}(|x|), && d \geqslant 2 \end{aligned} \tag{4.13}$$

Notice that $\tilde{Q}(|x|) = \frac{1}{2}(M_\varepsilon - 1)|x|^2 + \hat{Q}(|x|)$ for $d \geqslant 2$. Let us define for each $m \in \mathbb{N}$

$$\begin{aligned} &P_m(x) = P(0) + \tfrac{1}{2} M_\varepsilon |x|^2 + \int_0^x \hat{P}'(g_m(y))\, dy, && d = 1 \\ &P_m(x) = P(0) + b \cdot x + \tfrac{1}{2} M_\varepsilon |x|^2 + \int_0^{|x|} \hat{Q}'(g_m(r))\, dr, && d \geqslant 2 \end{aligned} \tag{4.14}$$

As in (2.8) and (2.9), we write for $\omega = (\omega_i)_{i \in \Lambda_n} \in E^{\Lambda_n}$ and $\bar{\xi} = (\bar{\xi}_i)_{i \in \mathbb{Z}^\nu} \in \mathscr{S}$

$$\begin{aligned} V_m(\omega_{\Lambda_n}) &= \sum_{i \in \Lambda_n} \int_0^1 P_m(\omega_i(\tau))\, d\tau + \sum_{i, j \in \Lambda_n} \int_0^1 U(\omega_i(\tau), \omega_j(\tau); |i-j|)\, d\tau \\ W(\omega_{\Lambda_n}, \bar{\xi}_{\Lambda_n^c}) &= \sum_{i \in \Lambda_n, j \in \Lambda_n^c} \int_0^1 V(\omega_i(\tau), \bar{\xi}_j(\tau); |i-j|)\, d\tau \end{aligned} \tag{4.15}$$

For given $\bar{\xi} \in \mathscr{S}$ and $m, n \in \mathbb{N}$, we define a probability measure $\mu_m^{(n)}$ on E^{Λ_n} by

$$d\mu_m^{(n)}(\omega_{\Lambda_n}) = \frac{1}{Z_m^{(n)}} \exp\{-V_m(\omega_{\Lambda_n}) - W(\omega_{\Lambda_n}, \bar{\xi}_{\Lambda^c})\}\, \lambda(d\omega_{\Lambda_n}) \tag{4.16}$$

where $Z_m^{(n)}$ is the normalization factor. For given $\Lambda_n \in \mathscr{C}$, let $\{b_m^{(n)} : m \in \mathbb{N}\}$ be the sequence of maps $b_m^{(n)}: \mathsf{H}_-^{(n)} \to \mathsf{H}_-^{(n)}$, $m \in \mathbb{N}$, defined by

$$b_m^{(n)}(\omega) = \beta_1^{(n)}(\omega) + b_{m,2}^{(n,1)}(\omega) + \beta_2^{(n,2)}(\omega), \qquad \omega = (\omega_i) \in \mathsf{H}_-^{(n)} \tag{4.17}$$

where $\beta_1^{(n)}$ and $\beta_2^{(n,2)}$ have been defined by (4.12), and $b_{m,2}^{(n,1)}$ is defined by

$$(b_{m,2}^{(n,1)}(\omega))_i = \begin{cases} -A^{-1}((M_\varepsilon - 1)\,\omega_i + \hat{V}'(g_n(\omega_i))), & d=1 \\ -A^{-1}\left(b + (M_\varepsilon - 1)\,\omega_i + \hat{Q}'(g_n(\omega_i))\dfrac{\omega_i}{|\omega_i|}\right), & d \geqslant 2 \end{cases} \tag{4.18}$$

By a direct calculation, it can be proved that the maps $b_m^{(n)}$ defined in (4.17)–(4.18) are the logarithmic derivatives of the measure $\mu_m^{(n)}$, $m \in \mathbb{N}$, on E^{Λ_n}. See the proof of Lemma 4.2 (a) below.

Let us denote by $H_{\mu_m^{(n)}}$ the Dirichlet operators for the measures $\mu_m^{(n)}$ (with respect to the rigging (4.11)). The following is a result corresponding to Lemma 5.1 of ref. 37.

Lemma 4.2. Suppose that the assumptions in Theorem 2.10 are satisfied. For each $n, m \in \mathbb{N}$, the following results hold:

(a) $H_{\mu_m^{(n)}}$ is essentially self-adjoint on $\mathscr{F}C_b^\infty(\mathsf{H}_-^{(n)})$.

(b) Let $T_t^{(n,m)} = \exp(-tH_{\mu_m^{(n)}})$ be the corresponding semi-group on $L^2(\mathsf{H}_-^{(n)}, \mu_m^{(n)})$. Then $T_t^{(n,m)}$, $t \in \mathbb{R}_+$, forms a positive preserving semi-group from $C_{\mathrm{pol}}^2(\mathsf{H}_-^{(n)})$ into itself.

(c) For any $\varepsilon > 0$ such that $M - M' - \varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for $m \geqslant N(\varepsilon)$ the measure $\mu_m^{(n)}$ satisfies the log-Sobolev inequality with Sobolev coefficient $c_{\mu_m^{(n)}} = \tilde{\lambda}(\varepsilon)^{-1}$ uniformly in n and m, where $\tilde{\lambda}(\varepsilon) = \min\{1, M - M' - \varepsilon\}$.

Proof. (a) It follows from (4.14) that

$$P'_m(x) = \hat{P}'(g_m(x)) + M_\varepsilon x, \qquad d=1$$

$$P'_m(x) = \hat{Q}'(g_m(|x|))\, x/|x| + M_\varepsilon x + b, \qquad d \geqslant 2$$

Thus a calculation shows that for each $n, m \in \mathbb{N}$ the map $b_m^{(n)}$ defined in (4.17)–(4.18) is the logarithmic derivative of the measure $\mu_m^{(n)}$ defined in (4.16). Using the method similar to that employed in the proof of Lemma 3.2, one can show that the conditions (3.1) and (3.2) corresponding to $\beta_m^{(n)} \equiv b_m^{(n)}$ and $\mu_m^{(n)}$ are satisfied. By the definition of $b_m^{(n)}$ in (4.17) and (4.18), and the method used in Lemma 3.4, it is easy to show that the conditions (i) and (ii) of Theorem 3.1 for $b_m^{(n)}$ are satisfied. Notice that $\mathsf{H}_- = L^2([0,1], d\tau)$. Using Assumptions 2.1 (b) and (d), and the method used in the proof of Lemma 3.5 (b), one can show that for each $n, m \in \mathbb{N}$ there exists a constant $c(n,m)$ such that the bound

$$((\phi, (b_m^{(n)})'(\omega)\,\phi))_{\mathsf{H}_-^{(n)}} \leqslant c(n,m)\,\|\phi\|_{\mathsf{H}_-^{(n)}}, \qquad \omega \in \mathsf{H}_-^{(n)}$$

holds for any $\phi \in \mathsf{H}_-^{(n)}$ uniformly in $\omega \in \mathsf{H}_-^{(n)}$. By Theorem 1 and Lemma 6 of ref. 7, the above results imply the part (a) of the lemma.

(b) This follows from Lemma 4 of ref. 7.

(c) Consider the case for $d \geqslant 2$. The case for $d = 1$ can be handled in the similar manner. From the definition of $b_{m,2}^{(n,1)}$ in (4.18), it follows that for any $\phi = (\phi_i)_{i \in \Lambda_n} \in \mathsf{H}_0^{(n)}$ and $\omega = (\omega_i)_{i \in \Lambda_n} \in \mathsf{H}_-^{(n)}$

$$(\phi_i, ((b_{m,2}^{(n,1)})'(\omega)\,\phi)_i)_0 = -(M_\varepsilon - 1)\,|\phi_i|_{L^2}^2 - (\phi_i, \hat{R}_m(\omega_i)\,\phi_i)_{L^2} \qquad (4.19)$$

where

$$\hat{R}_m(x) = \hat{Q}''(g_m(|x|))\,g_m'(|x|)\,\vec{x}\vec{x}/|x|^2 + \hat{Q}'(g_m(|x|))(\mathbf{1} - \vec{x}\vec{x}/|x|^2)/|x|$$

See the expression $\tilde{R}_n(x)$ in (3.47). Since Hess. $Q(|x|) \geqslant M\mathbf{1}$, Hess. $\hat{Q}(|x|) \geqslant \varepsilon \mathbf{1}$. Thus we may assume that there exists $R > 0$ such that

$$\hat{Q}'(|x|) \geqslant 0 \qquad \text{and} \qquad \hat{Q}''(|x|) \geqslant 0, \qquad \text{if} \quad |x| > R$$

By the argument similar to that used in the proof of the inequality (3.62), we prove that for any $x, y \in \mathbb{R}^d$

$$(y, \hat{R}_m(x)\,y) \geqslant 0$$

if $m \geqslant N(\varepsilon)$ for some $N(\varepsilon) \in \mathbb{N}$. It follows from (4.19) and the above bound that

$$-(\phi_i, (b_{m,2}^{(n,1)})'(\omega)\,\phi_i)_0 \geqslant (M_\varepsilon - 1)\,|\phi_i|_{L^2}^2$$

The above bounds imply that for any $m \geqslant N(\varepsilon)$

$$-((\phi, (b_{m,2}^{(n,1)})'(\omega)\,\phi))_{\mathsf{H}_0^{(n)}} \geqslant (M_\varepsilon - 1) \sum_{i \in \Lambda_n} |\phi_i|_{L^2}^2 \qquad (4.20)$$

uniformly in $m, n \in \mathbb{N}$.

On the other hand it follows from the definition in (4.12) that

$$-((\phi, (\beta_1^{(n)})'(\omega)\,\phi))_{\mathsf{H}_0^{(n)}} = \sum_{i \in \Lambda_n} (\phi_i, (-\Delta_p + \mathbf{1})\,\phi_i)_{L^2} \qquad (4.21)$$

Using the definition of $\beta_2^{(n,2)}$ in (4.12) and Assumption 2.6 (d), we obtain that

$$
\begin{aligned}
&|((\phi, (\beta_2^{(n,2)})'(\omega, \bar{\xi})\,\phi))_{\mathrm{H}_n^{(n)}}| \\
&\quad = \left| \sum_{i \in \Lambda_n} \left(\phi_i, \left\{ \sum_{j \in \Lambda_n :\, j \neq i} \partial^i \partial^i U(\omega_i, \omega_j)\,\phi_i + \partial^j \partial^i U(\omega_i, \omega_j)\,\phi_j \right.\right.\right. \\
&\qquad \left.\left.\left. + \sum_{j \in \Lambda_n^c} \partial^i \partial^i U(\omega_i, \bar{\xi}_j)\,\phi_i \right\} \right)_{L^2} \right| \\
&\quad \leqslant d \sum_{i \in \Lambda_n} |\phi_i|_{L^2} \left\{ \sum_{j \in \Lambda_n} \Psi(|i-j|)(|\phi_i|_{L^2} + |\phi_j|_{L^2}) + \sum_{j \in \Lambda_n^c} \Psi(|i-j|)\,|\phi_i|_{L^2} \right\} \\
&\quad \leqslant M' \sum_{i \in \Lambda_n} |\phi_i|_{L^2}^2 \qquad (4.22)
\end{aligned}
$$

uniformly in $m, n \in \mathbb{N}$ and $\bar{\xi} \in \mathscr{S}$. The bounds in (4.20)–(4.22) imply that

$$
\begin{aligned}
-((\phi, (b_m^{(n)})'(\omega, \bar{\xi})\,\phi))_{\mathrm{H}_0^{(n)}} &\geqslant \sum_{i \in \Lambda_n} (\phi_i, (-\Delta_p + M_\varepsilon - M')\,\phi_i)_{L^2} \\
&\geqslant \tilde{\lambda}(\varepsilon)\, \|\phi\|_{\mathrm{H}_0^{(n)}}
\end{aligned}
$$

uniformly in $m, n \in \mathbb{N}$ and $\bar{\xi} \in \mathscr{S}$. The above implies that the measure $\mu_m^{(n)}$ is uniformly log-concave. The part (c) of the lemma follows from the part (a) of the lemma and the method used in the proof of Theorem 3 of ref. 7. For the details, we refer to ref. 7. This proves the lemma completely. ∎

We now produce the proof of Theorem 2.12.

Proof of Theorem 2.12. As stated before, it suffices to show the theorem for $\gamma = 0$. Using (4.13), (4.15), and Assumption 2.1 (c), it can be checked that for given $n \in \mathbb{N}$ and $\bar{\xi} \in \Omega_{\log}$ there exists a constant $\varepsilon > 0$ and $c(n, \bar{\xi})$ such that

$$
\exp\{-V_m(\omega_{\Lambda_n}) - W(\omega_{\Lambda_n}, \bar{\xi}_{\Lambda^c})\} \leqslant \exp\left\{-\tfrac{1}{2}(M_\varepsilon - M') \sum_{i \in \Lambda_n} |\omega_i|^2 + c(\bar{\xi}, n)\right\}
$$

Since $M_\varepsilon - M' > 0$, the right hand side of the above is λ^{Λ_n}-integrable. Also, by Lemma 3.3 (a) and (e),

$$
\sum_{i \in \Lambda_n} P_m(\omega_i) \to \sum_{i \in \Lambda_n} P(\omega_i) \qquad \text{as} \quad m \to \infty
$$

Thus we use the Lebesgue dominated convergence theorem to show that for each $n \in \mathbb{N}$, $\mu_m^{(n)}$ converges to $\mu^{(n)}$ weakly as $m \to \infty$. By Lemma 4.2 (c), we conclude that for each n, the local Gibbs measure $\mu^{(n)} = \gamma_{\Lambda_n}^{\Phi}(\cdot \mid \bar{\xi})$

satisfies the log-Sobolev inequality with Sobolev coefficient $\tilde{\lambda}(\varepsilon)^{-1}$. Since $\varepsilon > 0$ is arbitrary, we see that the log-Sobolev inequality holds with a Sobolev coefficient $c_\mu = \tilde{\lambda}^{-1}$, $\tilde{\lambda} = \min\{1, M - M'\}$. Since for $f \in \mathscr{F}_{\text{loc}} C_b^\infty(\mathscr{H}_-)$, $\mu^{(n)}(f) \to \mu(f)$ as $n \to \infty$, the same conclusion holds for μ. This completes the proof of Theorem 2.12. ∎

5. IMPROVEMENTS AND CONCLUDING REMARKS

It would be worth to give comments on some conditions in Assumption 2.6. The requirements of the spherical symmentricity of one-body interactions for $d \geqslant 2$ (Assumption 2.6 (c)) and the exponential decay property of two-body interactions (Assumption 2.6 (d)) can be relaxed in some situations.

Let us first consider a possibility of relaxing Assumption 2.6 (c).[37] For $d \geqslant 2$ let us assume that the one-body interaction $P \in C^3(\mathbb{R}^d; \mathbb{R})$ can be written as

$$P(x) = Q(|x|) + F(x), \qquad x \in \mathbb{R}^d \tag{5.1}$$

where F is a C^3-function satisfying the following bounds: there exists a constant $K > 0$ such that for any $x \in \mathbb{R}^d$

$$\begin{aligned} |\partial F(x)|_{\mathbb{R}^d} &\leqslant K(1 + |x|) \\ \|\text{Hess. } F(x)\|_{\mathscr{L}(\mathbb{R}^d, \mathbb{R}^d)} &\leqslant K \end{aligned} \tag{5.2}$$

and $F^{(3)}(x)$ is bounded (in the oeprator norm) uniformly in $x \in \mathbb{R}^d$, where $F^{(3)}$ is the third order derivative of F.

Theorem 5.1. Instead of Assumption 2.6 (c), let us assume that the one-body interaction P satisfies (5.1) and (5.2) for $d \geqslant 2$. Then Theorem 2.7 and Corollary 2.11 still hold.

Proof. Recall the definition b_n in (3.37)–(3.39) and (3.41). For $n \in \mathbb{N}$ and $i \in \mathbb{Z}^\nu$, we replace $b^{(1)}_{2,n,i}(\xi_i)$ in (3.41) by

$$\begin{aligned} b^{(1)}_{2,n,i}(\xi_i) = &- A^{-(1-\gamma)}\{G_n \tilde{Q}'(g_n(|G_n \xi_i|))(G_n \xi_i / |G_n \xi_i|)\} \\ &- A^{-(1-\gamma)}\{G_n(\partial F(G_n \xi_i))\} \end{aligned}$$

Due to (5.1) and (5.2), it is easy to check that Lemma 3.4–Lemma 3.7 remain true. This implies Theorem 5.1. We leave the details to the reader. ∎

Next, we consider polynomially bounded one-body interactions for which we can relax the exponential decay property of two-body interactions in Assumption 2.6 (d). Assume that the one-body interaction P belongs to $C^3_{\text{pol}}(\mathbb{R}^d, \mathbb{R})$. Thus, there exist $q \in \mathbb{N}$ and $M > 0$ such that the bound

$$|P(x)| + \|\partial P(x)\|_{\mathbb{R}^d} + \|\text{Hess. } P(x)\|_{\mathscr{L}(\mathbb{R}^d, \mathbb{R}^d)} \leqslant M(1 + |x|)^q, \qquad x \in \mathbb{R}^d \quad (5.3)$$

Instead of l_s, $s \in \{+, 0, -\}$, defined in (2.16), we introduce Hilbert spaces on the space of real-valued sequences as follows: for a given (fixed) positive real number p with $p > d$, put

$$l_s := \left\{ (a_i)_{i \in \mathbb{Z}^\nu} : \sum_{i \in \mathbb{Z}^\nu} (|i| + 1)^{sp} |a_i|^2 < \infty \right\}, \qquad s \in \{+, 0, -\} \quad (5.4)$$

For each $\gamma \in [0, 1]$ and $s \in \{+, 0, -\}$, let $\mathsf{H}_s^{(\gamma)}$ be the Hilbert spaces defined in (2.14) and let $\mathscr{H}_s^{(\gamma)} = \mathsf{H}_s^{(\gamma)} \otimes l_s$. Thus, as in (2.17)

$$\mathscr{H}_+^{(\gamma)} \subset \mathscr{H}_0^{(\gamma)} \subset \mathscr{H}_-^{(\gamma)} \quad (5.5)$$

is a rigging of $\mathscr{H}_0^{(\gamma)}$ by $\mathscr{H}_+^{(\gamma)}$ and $\mathscr{H}_-^{(\gamma)}$. We also define $\mathscr{K}_+^{(\gamma)}$ and $\mathscr{K}_-^{(\gamma)}$ analogously (cf. (3.20)). For any $\gamma \in [0, 1]$ and Gibbs measure μ, let $\mathscr{E}_\mu^{(\gamma)}$ and $H_\mu^{(\gamma)}$ be the Dirichlet form and associated Dirichlet operator with respect to the rigging (5.5).

Let Ψ be the function in Assumption 2.1 (b). Assume that there exist a positive real number p with $p > d$ and $K > 0$ such that

$$\Psi(r) \leqslant K(r + 1)^{-2p} \quad (5.6)$$

In (5.4) we choose the positive real number $p > d$ such that the bound (5.6) holds.

Theorem 5.2. Instead of Assumption 2.1 (b) and Assumption 2.6 (a), let P satisfy the bound (5.3). In addition, assume that the function Ψ in Assumption 2.1 (c) and Assumption 2.6 (d) satisfies the bound (5.6). Then, all the results listed in Section 2.2 still hold with respect to the riggings in (5.4) and (5.5).

Proof. Let $\gamma \in [0, 1]$ be given and we suppress γ in the notation. Notice that $\Omega_{\log} \subset \mathscr{H}_-$. We first show that $\|\beta(\xi)\|_- < \infty$ for any $\xi \in \Omega_{\log}$ and so $\|\beta\|_-$ is defined μ-a.e. It follows from (5.4) that for any $\xi = (\xi_i) \in \Omega_{\log}$

$$\|\beta(\xi)\|_-^2 = \sum_{i \in \mathbb{Z}^\nu} (|i| + 1)^{-p} |\beta_i(\xi)|_-^2 \quad (5.7)$$

where $|\beta_i(\xi)|_-$ satisfies the bound in (3.22). Consider the second term in the right hand side of (3.22). If $\xi=(\xi_i)\in\Omega_{\log}$, there exists $N\in\mathbb{N}$ such that for any $0\neq i\in\mathbb{Z}^\nu$, $|\xi_i|_u\leqslant N\log(|i|+1)$. Thus, by (5.3) we have that

$$|\partial\tilde{P}(\xi_i)|_{L^2}\leqslant M(1+|\xi_i|_u)^q\leqslant M(1+N\log(|i|+1))^q$$

By (5.6) and Assumption 2.1 (d), the last term in the right hand side of (3.22) is bounded by $M'(N\log(|i|+1))$. By (3.22) and (5.7), the above results imply that $\|\beta(\xi)\|_-<\infty$.

The estimates in (3.24), (3.26), and (3.27) imply that $|\beta|_-^4\in L^2(\mu)$. Let $\{b_n\}$ be the sequence of maps defined as in (3.37)–(3.41). We claim that all the results in Lemma 3.4–Lemma 3.7 also hold in this setting. Notice that for given $p\in\mathbb{N}$ there exists a constant $M(p)>0$ such that for any $i, j\in\mathbb{Z}^\nu$ the bound

$$(|i|+1)^{-p/2}(|i-j|+1)^{-2p}(|j|+1)^{p/2}\leqslant M(p)(|i-j|+1)^{-p} \tag{5.8}$$

The above is the bound corresponding to that in (3.43). In order to give the basic idea of the proof of our claim, consider Lemma 3.6. Let us prove that the bound (3.60). By using (5.6), (5.8), and the method used in (3.59)–(3.60), we obtain that for any $\xi\in\mathscr{H}_0$ and $h\in\mathscr{K}_-$

$$\begin{aligned}((h, b_{2,n}^{(2)\prime}(\xi)\,h))_{\mathscr{K}_-}\leqslant K\Bigg\{\|h\|_{\mathscr{K}_-}^2+\sum_{i\in\mathbb{Z}^\nu}(|i|+1)^{-p/2}\,|h_i|_0\\ \times\Bigg\{\sum_{j\neq i}(|i-j|+1)^{-p}\,(|j|+1)^{-p/2}\,|h_j|_0\Bigg\}\Bigg\}\\ \leqslant C'\,\|h\|_{\mathscr{K}_-}^2\end{aligned}$$

for some constant $C'>0$. It is obvious that the methods used in the proofs of Lemma 3.4–Lemma 3.7 can be applied to prove our claim. Also, the methods used in Section 4 can be applied to the new setting. We leave the details to the reader. ▮

In Theorem 2.12, we have stated the log-Sobolev inequality with a Sobolev coefficient $c_\mu=\tilde{\lambda}^{-1}$, $\tilde{\lambda}=\min\{1, M-M'\}$, uniformly in $\gamma\in[0,1]$. We may take $c_\mu=(M-M')^{-1}$ for $\gamma=1$. As in ref. 9, we can proceed with $A=-\Delta_p+M\mathbf{1}$ instead of $A=-\Delta_p+\mathbf{1}$ in the definition of rigged Hilbert spaces in (2.13)–(2.14). Then, we have $\tilde{\lambda}(\varepsilon)=(M_\varepsilon-M')/M$ in Lemma 4.2 (c) for $\gamma=0$ (see the inequality below (4.22)). Using the same method used in the proof of Theorem 2.12, we have the following log-Sobolev inequality for $\gamma=0$:

$$\int_{\mathscr{H}_-} |f(\xi)|^2 \log |f(\xi)| \, d\mu(\xi)$$

$$\leqslant \frac{M}{M-M'} \int_{\mathscr{H}_-} (\|\nabla^{(0)} f(\xi)\|_0^{(0)})^2 \, d\mu(\xi) + \|f\|_{L^2(\mu)}^2 \log \|f\|_{L^2(\mu)} \tag{5.9}$$

On the other hand, with a newly defined A, we have the following inequality:

$$(\|\nabla^{(0)} f(\xi)\|_0^{(0)})^2 \leqslant \frac{1}{M} (\|\nabla^{(1)} f(\xi)\|_0^{(1)})^2 \tag{5.10}$$

Our claim now follows from (5.9) and (5.10).

It would be worthwhile mentioning the stochastic dynamics related to the Dirichlet forms. Let us use the same notation $(\mathscr{E}_\mu^{(\gamma)}, D(\mathscr{E}_\mu^{(\gamma)}))$ for the closure of the pre-Dirichlet form of (2.25). In ref. 32, we have shown that $(\mathscr{E}_\mu^{(0)}, D(\mathscr{E}_\mu^{(0)}))$ is a quasi regular Dirichlet form.[39] The method can be applied to show that $(\mathscr{E}_\mu^{(\gamma)}, D(\mathscr{E}_\mu^{(\gamma)}))$ is a quasi regular Dirichlet form for any $\gamma \in [0, 1]$. In fact, it follows from a general theory (see ref. 39, Chapter IV, Section 3) since the condition (3.1) of ref. 39, page 170, holds trivially in our case. Since $(\mathscr{E}_\mu^{(\gamma)}, D(\mathscr{E}_\mu^{(\gamma)}))$ has also a local property,[34] there exists a diffusion process $\mathbb{M} := (\tilde{\Omega}, \tilde{F}, (X_t)_{t \geqslant 0}, (P_\xi)_{\xi \in \mathscr{H}_-^{(\gamma)}})$ which is properly associated with $(\mathscr{E}_\mu^{(\gamma)}, D(\mathscr{E}_\mu^{(\gamma)}))$, i.e., for each bounded $u \in L^2(\mu)$,

$$(T_t u)(\xi) = \int_{\tilde{\Omega}} u(X_t) \, dP_\xi \qquad \text{for} \quad \mu\text{-a.a. } \xi \in \mathscr{H}_-^{(\gamma)} \tag{5.11}$$

Furthermore, since the embeddings $\mathscr{H}_+^{(\gamma)} \subset \mathscr{H}_0^{(\gamma)} \subset \mathscr{H}_-^{(\gamma)}$ are Hilbert–Schmidt and the conditions (C.3) and (C.4) of ref. 39, Chapter V, Section 3, are satisfied from Lemma 3.2 of this paper, we see then from Proposition 2.5 and Theorem 3.1 of ref. 39, Chapter V, that there exists a $\mathscr{H}_-^{(\gamma)}$-valued Brownian motion $(W_t)_{t \geqslant 0}$ over $\mathscr{H}_0^{(\gamma)}$ such that the process $\mathbb{M}$ weakly solves the stochastic differential equation of the following type:

$$\begin{aligned} dX_t &= dW_t + \tfrac{1}{2} \beta^{(\gamma)}(X_t) \, dt \\ X_0 &= \xi \qquad \text{under } P_\xi \end{aligned} \tag{5.12}$$

That is, there exists a countable orthonormal system $K^{(\gamma)} (\subset \mathscr{H}_+^{(\gamma)})$ of $\mathscr{H}_0^{(\gamma)}$ such that

$$u_k(X_t) - u_k(X_0) = W_t^k + \tfrac{1}{2} \int_0^t \beta_k^{(\gamma)}(X_s) \, ds, \qquad t \geqslant 0, \quad k \in K^{(\gamma)} \tag{5.13}$$

holds, where $u_k(\xi) := {}_{H_+^{(\gamma)}}\langle\langle k, \xi \rangle\rangle_{H_-^{(\gamma)}}$, $\xi \in \mathscr{H}_-^{(\gamma)}$, and for all $\xi \in \mathscr{H}_-^{(\gamma)}$ except a capacity zero set, $(W_t^k, \mathscr{F}_t, P_\xi)_{t \geq 0}$ is a one-dimensional Brownian motion starting at zero for all $k \in K^{(\gamma)}$. Moreover, for each $k \in K^{(\gamma)}$, ${}_{H_+^{(\gamma)}}\langle\langle k, W_t \rangle\rangle_{H_-^{(\gamma)}} = W_t^k$, $t \geq 0$, P_ξ-a.s., for all $\xi \in \mathscr{H}_-^{(\gamma)}$ except a capacity zero set. For the details we refer to ref. 39.

The stochastic dynamics (5.12) and the Gibbs states for quantum unbounded spin systems have been also constructed as an infinite volume limit of the corresponding finite volume cut-off systems by S. Albeverio, Yu. G. Kondratiev, and T. V. Tsikalenko in ref. 10. Under proper conditions, they have shown the existence of a unique solution to (5.12) (ibid, Theorem 4), the ergodic property of the solution Markov process (ibid, Theorem 5), and the existence of a unique invariant (Gibbs) measure (ibid, Theorem 6). For the details we refer to ref. 10. Thus, this paper can be viewed as a complement of ref. 10.

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